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On Robust alternatives to $\overline{x}$ and S control charts: A minimum distance approach

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ABSTRACT

In this article, we propose the application of a class of weighted likelihood estimators (WLE) based on the minimum distance approach to the control chart scenario for performing statistical process control. The WLEs are known to be highly robust under model misspecification and presence of outliers in the data, but they are also asymptotically fully efficient in parametric models; moreover, in large samples they behave like maximum likelihood estimators at the true model; Markatou et al., 1997, 1998 and Basu et al. (2011). Our proposal aims at building reliable, resistant, robust and highly efficient control limits for control charts using the WLEs of the process parameters; the subgroup statistics which are then to be plotted on the charts should be stable in the presence of outliers, so that any slight (assignable) shift in the process could be easily and rapidly detected. Earlier work in this area motivated towards the goal of building robust control charts were based on the robust statistics for the location and for the scale; Rocke (1989) and Abu-Shawiesh, 2008, 2009). Many of these methods are robust under the model misspecifiction and presence of outliers in the data; however all the estimators involved in these procedures are deficient at the model, some of them severely. Hence, there is no reason to rely more on these statistics if one has a more efficient and robust option offered by the WLE. The simulation and comparative study of the proposed method are done and it is observed that in terms of overall performance of the control charts, the proposed method is competitive or better than the available techniques; moreover if the data are pure and represent the normal model correctly, the proposed method behaves like the classical technique of Shewhart (1931). To give a clear illustration of the performance of the proposed technique some real data examples are included.

1. Introduction

In statistical quality control, the technique of control charts was introduced by Shewhart (1931, 1939); it is widely used to detect shift or variation in an industrial process by differentiating between two causes of variation: random and assignable. The standard procedure prescribed in literature for the development of control charts for an industrial process submitted to statistical quality control is to take at least 20 subgroup samples each of size about 5 rationally; compute sample mean and sample standard deviations of each subgroup and construct control limits using the mean of the subgroup sample means and sample standard deviations in a manner that if the process is in statistical control, then the subgroup statistics falling beyond these limits should be ascribed only to random variations. This charting procedure assumes normality and stability in the concerned process. If these assumptions hold good, a quality engineer may easily and quickly detect any assignable variation in the process with the help of these charts and may send an alarm to the manager for possible corrective measures which in turn minimize the losses due to scraps to a great extent. Duncan (1953) and Cowden (1957).

Shewhart charts perform poorly under the model misspecification and presence of outliers in the data, since control limits for the charts are based on statistics sample mean and sample standard deviation. These statistics are known to be highly non-robust and nonresistant even under slight deviations from the model assumptions and a small contamination in the data. Hence it becomes very difficult to detect any intermittent behavior in the process using the control limits based on these statistics. The method we are going to propose here aims at building resistant and robust control limits for the control charts; the subgroup statistics which are to be plotted on the charts should be sensitive to outliers, so that prompt and early detection of any systematic variation in the process could be easily possible. The proposed technique can ably ameliorate the risk engendered by the possibility of the model misspecification and contamination in the data.

Past works in this field motivated towards the goal of building robust control charts were based on the robust statistics for the location (*e.g.* sample median, trimmed mean etc) and for the scale (*e.g.* sample median absolute deviation, sample inter-quartile range etc). In this regard Ferrell (1953) introduced a method to determine control limits using the median of subgroup mid-ranges and the

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median of subgroup ranges; Langenberg and Iglewicz (1986) established a method for mean and range charts with control limits determined by the trimmed mean of the subgroup means and the trimmed mean of the subgroup ranges; Iglewicz and Hoaglin (1987) and White and Schroeder (1987) introduced the method of plotting subgroup boxplots based on the subgroup median and the subgroup inter-quartile range (IQR); Rocke (1989) proposed the methods of constructing control charts based on different combination of subgroup statistics for location (*e.g.* mean, median) and for scale (*e.g.* range and IQR) with control limits calculated from the trimmed mean of subgroup means; Rocke (1992) proposed the method of construction control charts for mean and range with control limits determined from the average subgroup IQR; Abu-Shawiesh (2009) proposed method of constructing control chart for mean based on subgroup median and MAD, Abu-Shawiesh (2008) introduced *S* control chart based on subgroup MAD. The primary motivation in most of these papers has been to find a quick and easy solution to the robustness problem of the subgroup statistics. Among all the robust statistics used in these papers for the location and scale parameters, subgroup median and subgroup MAD keep maximal breakdown property, but these two are also highly inefficient compared to the sample mean and sample standard deviation respectively. However the control charts introduced by Ferrell (1953) and Langenberg and Iglewicz (1986) are expected to perform better than the classical method under contamination in the data; also the methods of control charting proposed by Rocke (1989, 1992) have good properties in terms of robustness and may perform better than any earlier techniques under deviation from the model assumptions and impurities in the data. Similar points are observed in the techniques suggested by Abu-Shawiesh (2008, 2009). It appears, however all these methods in the literature only provide secondary consideration to the issue of the efficiency of the method when the observations are actually realized from the pure data without contamination, thus sacrificing some reliability at the true model.

The WLEs come as solutions of weighted likelihood estimating equations constructed based on the minimum distance approach whereas the weights are so constructed as to discount the effect of the observations incompatible with the rest of the data. These estimators are known to be highly robust under the model misspecification and contamination in the data. Also under the normality assumptions they are asymptotically efficient estimators of the concerned parameters. The corresponding influence function comes out to be unbounded, as would be necessary for full asymptotic efficiency; moreover this method behaves like the maximum likelihood at the true model. See Basu and Lindsay (1994) and Basu et al (2011) for a comprehensive description.

We shall use a weighted likelihood estimating equation approach which follows in a natural way from some density-based minimum distance estimating equations. The weight for an observation ranges from 0 to 1 depending on the degree of deviation of the corresponding observation from the model in relation to the rest of the data. For example when the general trend in the data wholly follows a model pattern, the observations may all get weights close to 1. On the other hand when the majority of the data follow the model, one (or a few) discrepant point(s) may get weight(s) close to zero. Solving the weighted likelihood equations, which are a random linear combinations of likelihood score functions of observations equated to zero, produce the WLEs. This approach was initially introduced by Green (1984); later work in this area include Markatou *et al.* (1997, 1998) and Basu and Lindsay (2004).

Given a data point x in the sample space, lets us consider a (relative) residual function $\delta(x; F_\theta; F_n)$ where F_θ is the distribution function of the model, and F_n is the empirical distribution function. Let $w(.)$ be the weight function based on the residual δ and a tuning parameter say c. Now, one can have the WLEs $\hat{\theta}_w$ as the solution of the weighted likelihood equations

$$\sum_{i=1}^{n} w\left(c, \delta\left(x_i, F_\theta, F_n\right)\right) u_\theta\left(x_i\right) = 0 \qquad (1)$$

where $u_\theta(x)$ is the likelihood score corresponding to the observation x.

The simple form of equation (1) provides a natural algorithm based on iterative reweighting. And at the end, the final fitted weights indicate which of those points were downweighted in the final solution relative to the MLE. It should be noted that the (relative) residual function is constructed in such a way that it can take values in $[-1, \infty)$ while the weight function is uni-modal and takes value in [0, 1].

Evidently, the weight function plays a vital role in the construction of the weighted likelihood procedure. The decision about the weight function is a very crucial step towards the process of finding the robust and highly efficient weighted likelihood estimators. So, the weight function should be carefully chosen which can serve the purpose at hand. The criterion for selection of the weight function is (i) it should be defined from $[-1, \infty)$ to [0,1], (ii) it should attain maximum value only at zero and decreases steadily in a smooth manner on either side of its domain.

Keeping these properties in mind three different weight functions are constructed as

$$w_1(c, \delta) = \exp(-c\delta^2), \quad (2)$$

$$w_2(c, \delta) = (1 + c\delta - c\log(1+\delta))^{-1}, \quad (3)$$

$$w_3(c, \delta) = ((1+\delta)\exp(-\delta))^c, \quad (4)$$

Here δ is the residual defined in equation (1) and c is called the tuning parameter of weight function, it controls the nature of weight function. These weight functions have also been used by other authors like Banerjee *et al.* (2014) and Biswas *et al.* (2015).

In this article, we implement the method of construction of weights based on model density and kernel density estimate discussed in Markatou *et al.* (1997, 1998). Apart from this method, we work with a weight function designed specially to serve the purpose.

We propose the method of constructing the control charts for $\overline{X}$ and S with control limits established based on subgroup WLEs while subgroup sample mean and sample standard deviation are to be plotted on control charts. Thus one can claim to have reliable, resistant, robust and highly efficient $\overline{X}$ and S control charts established on the basis of the proposed method.

2. Proposal

In this section, we discuss the WLE approach and different methods of construction of $\overline{X}$ and S charts. The WLE approach is illustrated through a method based on density divergences for finding the (relative) residual corresponding to a data with a typical weight function. Three control charting procedures for $\overline{X}$ and S are taken into account based on estimates of the process mean and process standard deviation used to establish the control limits are (i) the classical approach of Shewhart (1931) uses usual statistics mean of sample means and mean of sample standard deviations, (ii) the recently developed robust control charts by Abu-Shawiesh (2008, 2009) considers mean of sample medians and mean of sample median absolute deviations, and (iii) the proposed method mean of sample weighted means and mean of weighted standard deviations. The reason behind comparing performance of our proposed technique with the second procedure is simply due to this recently developed technique is claimed to perform better than the earlier development in robust charting procedures.

2.1 Weighted likelihood approach

This approach deals with methods for constructing weights with the right properties and solving the weighted likelihood equations (1) to generate the weighted likelihood estimators. We illustrate here use of this technique with three methods of creating the relative residuals and three weight functions which can be used to solve the weight equations.

2.1.1 Method 1

This method based on density based divergences is well known for simultaneously providing robust and efficient estimators of θ; for more details refer to Markatou *et al.* (1997, 1998) and Basu *et al.* (2011).

For a given observation x from a population density f_θ, one can construct the (relative) residual function as

$$\delta(x, F_\theta, F_n) = \delta(x, f_\theta^*, f_n^*) = \frac{f_n^*(x)}{f_\theta^*(x)} - 1 \quad (5)$$

where f_θ^* is smoothed model density and f_n^* is the smoothed kernel density estimate with the smoothing parameter h; the ordinary model density is denoted by f_θ.

Here both the data as well as the model are smoothed, this approach has certain advantages; over the usual method of smoothing the data alone, this is due to the fact that, if the model is true, then $\delta(x)$ converges to zero with probability one and this holds even in case of a fixed value of the smoothing parameter h.

2.1.2 Method 2

It is based on controlling the tail probabilities. A tuning parameter $p \le \frac{1}{2}$ denotes the proportion of outliers to the down-weighted on either tail. The relative residual at a given value compares the probabilities of getting more extreme observations in the given sample and under the model respectively.

Let for a given observation x; $F_n(x)$, $S_n(x)$, $F_\theta(x)$, and $S_\theta(x)$ be the empirical distribution function, the empirical survival function, the model distribution function and the model survival function respectively. Then one can construct the (relative) residual function as

$$\delta(x, F_\theta, F_n) = \begin{cases} \frac{F_n(x)}{F_\theta(x)} - 1 & if \;\; F_\theta(x) \le p, \\ 0 & if \;\; p < F_\theta(x) < 1-p \\ \frac{S_n(x)}{S_\theta(x)} - 1 & if \;\; F_\theta(x) \ge 1-p \end{cases} \quad (6)$$

The reason behind this typical residual function is due to the fact that generally outliers belong to either right-tail of the model for which $F_\theta(x) \ge 1-p$ or left-tail of the model for which $F_\theta(x) \le p$, and to ascertain the very reason the method itself compares the empirical survival function $S_n(x)$ with the survival function $S_\theta(x)$ of the model in case of the right-tail observations and the empirical distribution function $F_n(x)$ with the distribution function $F_\theta(x)$ in case of left-tail observations. The remaining (1– 2*p*) fraction of the observations which are lying in the central part of the assumed model and for which $p < F_\theta(x) < 1 - p$ are attached with weights of one. Hence, we can claim that this method is robust under the presence of outliers in the data set as it downweights 100*p*% observations on each tail if they are inconsistent with the model. However all the weights converge to one under the pure model and the method behaves like the maximum likelihood for large *n*.

Method 2 is based on the development in Biswas *et al.* (2015).

2.1.3 Method 3

This method is based on the probabilities of the different windows over the support. For each given observation, it compares the empirical probability of the data with the assumed model probability within the neighborhood of the data point under consideration. Here the neighborhood of a given observation *x* is defined by the window $[x - h, x + h]$ where $h = \frac{a}{\sqrt{n}}\sigma$ is termed as bandwidth parameter, *a* is a positive constant, and σ is the standard deviation of the assumed model estimated robustly from the data.

Given observation *x*, one can construct the residual function as

$$\delta(x, F_\theta, F_n) = \frac{F_n(x+h) - F_n(x-h)}{F_\theta(x+h) - F_\theta(x-h)} - 1 \qquad (7)$$

The neighborhood of a given observation *x* is the window $[x - h, x + h]$ where $h = \frac{a}{\sqrt{n}}\sigma$ is termed as bandwidth parameter, *a* is a positive constant, and σ is the standard deviation of the assumed model estimated robustly from the data.

This procedure of particular residual function can be thought of implementing the fact that the sparsity of the observations near and around an outlying data point as compared with the assumed model, could produce large values of the relative residual. To understand this technique more clearly, let us consider an outlier *x* present in the data set and the numerator of the residual function (7), which is actually the fraction of the observations present in $[x - h, x + h]$, comes out to be very unusual if compared with the denominator, which is the probability that an observation lies within range $[x - h, x + h]$ calculated based on the assumed model and this in turn drives the relative residual function to be either closer to – 1 or ∞. The situation when residual comes out be zero implies that distribution of the data set around the observation is consistent with the assumed model distribution around the same.

Method 3 is based on the development in Banerjee *et al.* (2014).

2.2 Control Chart for $\overline{X}$ and *S*

One can construct classical control charts for statistics *T* of process parameter θ by evaluating the control limits say upper control limit (UCL), central line (CL) and lower control limit (LCL) as given;

$$UCL = \mu_T + k\sigma_T;$$
$$CL = \mu_T;$$
$$LCL = \mu_T - k\sigma_T: \qquad (8)$$

Here $\mu_T = E(T)$ and $\sigma_T^2 = V(T)$ are the mean and variance of statistics T and function of location μ and scale parameter σ of the process respectively. And in case the standards of the process are not known, one can build the above control limits by replacing the process parameters with their unbiased estimates based on the data. The value of *k* should be determined in such a manner that if the process is in the state of statistical control, then the probability of a subgroup statistic falling beyond these limits is scarce and if this happens one may ascribe it to intermittent behavior in the process. Here we consider $k = 3$, which means that probability of a subgroup sending a out of control signal is only about 0.0027 under the normality assumptions.

Now, consider random sample of observations from a manufacturing process and let x_{ij} be the i^{th} observation in the j^{th} subgroup or sample in the data, where $i = 1, 2, ... n$ and $j = 1, 2, ..., m$.

The overall sample mean and standard deviation are

$$\bar{\bar{x}} = (mn)^{-1} \sum_{j=1}^{m} \sum_{i=1}^{n} x_{ij}$$

$$\bar{s} = m^{-1} \sum_{j=1}^{m} \left\{ n^{-1} \sum_{i=1}^{n} \left(x_{ij} - \bar{x}_j \right)^2 \right\}^{\frac{1}{2}}.$$

The overall sample MED and MAD are

$$\overline{MED} = m^{-1} \sum_{j=1}^{m} median\left(x_{ij}\right)$$

$$\overline{MAD} = m^{-1} \sum_{j=1}^{m} 1.48 \times median_i \left[x_{ij} - median_i\left(x_{ij}\right) \right]$$

Let us consider w_{ij} be the weight allocated to i^{th} observation in the j^{th} subgroup in the data, where i = 1, 2, ..., n, and j = 1, 2, ..., m. The overall sample weighted mean and standard deviation are

$$\bar{\bar{x}}_w = m^{-1}\sum_{j=1}^{m}\left(\sum_{i=1}^{n} w_{ij}\right)^{-1}\sum_{i=1}^{n} w_{ij}x_{ij}$$

$$\bar{s}_w = m^{-1}\sum_{j=1}^{m}\left\{\left(\sum_{i=1}^{n} w_{ij}\right)^{-1}\sum_{i=1}^{n} w_{ij}\left(x_{ij-\bar{x}\,j}\right)\right\}^{\frac{1}{2}}$$

2.2.1 Shewhart $\overline{X}$ and S Charts

The classical $\overline{X}$ and S charts are framed to detect the shift in the process average μ and the shift in the process standard deviation σ respectively. The classical control limits calculated based on the values of sample means and sample standard deviations are generally non-robust. And the presence of a single large outlier in any of the subgroups can cause these limits to be almost ineffective in detecting the assignable variation in the process. Montgomery, (2007).

Given the standards of the process μ and σ are unknown (as they usually will be). The standards are replaced by their unbiased estimates function of sample mean and sample standard deviation from the data in the equation (8).

The control limits for $\bar{x}$ chart are

$$\begin{cases} UCL = \bar{\bar{x}} + 3\dfrac{1}{\sqrt{n}\,c_n}\bar{s} \\ CL = \bar{\bar{x}}, \\ LCL = \bar{\bar{x}} - 3\dfrac{1}{\sqrt{n}\,c_n}\bar{s} \end{cases} \tag{9}$$

The control limits for s chart are

$$\begin{cases} UCL = \bar{s}\left(1+3\dfrac{1}{c_n}\sqrt{\dfrac{n-1}{n}-c_n^2}\right) \\ CL = \bar{s} \\ LCL = \max\left\{0,\ \bar{s}\left(1-3\dfrac{1}{c_n}\sqrt{\dfrac{n-1}{n}-c_n^2}\right)\right\}, \end{cases} \tag{10}$$

Here

$$c_n = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\sqrt{\frac{2}{n}}$$

2.2.2 Control Charts based on the Median and the MAD

The sample median (MED) and the median absolute deviation from the sample median (MAD) are the robust estimates of the parameters μ and σ, and could be used to construct the control limits for the charts. The control limits constructed based on MED and MAD are available in literature. (Abu-Shawiesh, 2008, 2009).

As the standards of the process μ and σ are unknown, the standards are replaced by their unbiased estimates functions of sample MED and sample MAD from the data in the equation (8).

The control limits for $\bar{x}$ chart are

$$\begin{cases} UCL = \overline{MED} + 3\dfrac{1}{\sqrt{n}}\, b_n\,\overline{MAD}\ , \\ CL = \overline{MED}\ , \\ LCL = \overline{MED} - 3\dfrac{1}{\sqrt{n}}\, b_n\ \overline{MAD}\ , \end{cases} \tag{11}$$

The control limits for s chart are

$$\begin{cases} UCL = c_n\, b_n\ \overline{MAD}\left(1+3\sqrt{\dfrac{n-1}{n}-c_n^2}\right), \\ CL = c_n\, b_n\ \overline{MAD}\ , \\ LCL = \max\left\{0,\ c_n\ b_n\ \overline{MAD}\left(1-3\sqrt{\dfrac{n-1}{n}-c_n^2}\right)\right\}, \end{cases} \tag{12}$$

Here

$$c_n = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\sqrt{\frac{2}{n}}$$

and b_n is called correction factor, deduced by Rousseeuw and Croux (1993).

Table 1 : The value of correction factor b_n for different subgroup sample size n

n	2	3	4	5	6	7	8	9	n > 9
b_n	1.196	1.495	1.363	1.206	1.200	1.140	1.129	1.107	$\frac{n}{n-0.8}$

2.2.3 Control charts based on the WLE approach

So far we have discussed two methods to construct the control limits for $\bar{x}$ and s charts. The first method is based on the MLEs of the process parameters while the second approach is based on robust estimates MED and MAD of the process parameters. The first one is not robust under the presence of outliers in the subgroup samples. The second one is robust but lacking in efficiency.

Table 2 : The control limits for chloride content data computed by Shewhart, MED-MAD, and WLE methods. The Method 1 with all the three weight functions used to calculate WLEs

	Classical Method	MED-MAD		WLE	
			$w_1(c = 0.1)$	$w_2(c = 0.3)$	$w_3(c = 0.3)$
Control limits for $\bar{x}$ chart					
LCL	0.383811	0.426673	0.410839	0.410734	0.410742
CL	0.562813	0.554688	0.559099	0.559075	0.559060
UCL	0.741814	0.682702	0.707359	0.707417	0.707378
Control limits for *s* chart					
LCL	0.003855	0.014322	0.003193	0.003194	0.003194
CL	0.126953	0.087103	0.105151	0.105208	0.105192
UCL	0.250052	0.167261	0.207109	0.207222	0.207189

Table 3 : The control limits for chloride content data computed by Shewhart, MED-MAD, and WLE methods. The method 2 with all the three weight functions used to calculate WLEs, where the tail probability p = 0.5

	Classical Method			MED-MAD	WLE
			$w_1(c = 0.001)$	$w_2(c = 0.015)$	$w_3(c = 0.015)$
Control limits for $\bar{x}$ chart					
LCL	0.383811	0.426673	0.413043	0.400252	0.400401
CL	0.562813	0.554688	0.559779	0.561844	0.561784
UCL	0.741814	0.682702	0.706515	0.723437	0.723167
Control limits for *s* chart					
LCL	0.003855	0.014322	0.00316	0.003480	0.003475
CL	0.126953	0.087103	0.10407	0.114606	0.114458
UCL	0.250052	0.167261	0.20498	0.225732	0.225440

Table 4: The control limits for chloride content data computed by Shewhart, MED-MAD, and WLE methods. The method 3 with all the three weight functions used to calculate WLEs, where the window constant *a* = 1

	Classical Method		MED-MAD	WLE	
			$w_1(c = 0.01)$	$w_2(c = 0.06)$	$w_3(c = 0.05)$
Control limits for $\bar{x}$ chart					
LCL	0.383811	0.426673	0.410702	0.400170	0.400035
CL	0.562813	0.554688	0.558946	0.561627	0.561822
UCL	0.741814	0.682702	0.707190	0.723083	0.723609
Control limits for *s* chart					
LCL	0.003855	0.014322	0.003192	0.003477	0.003484
CL	0.126953	0.087103	0.105139	0.114510	0.114744
UCL	0.250052	0.167261	0.207086	0.225542	0.226005

Now, we propose the method of constructing the control limits with the help of the WLEs of the process parameters. The WLEs are known to be robust, asymptotically fully efficient and unbiased under the normality assumptions; moreover they behave like the MLEs when the model is true; for more details refer to Basu *et al.* (2011). Hence by this approach, one can claim to have control limits which are robust and highly efficient. The control limits thus computed are capable to detect a slight assignable shift in the process quickly.

As the standards of the process μ and σ are unknown, the standards are replaced by their unbiased estimates functions of weighted sample mean and weighted sample standard deviation from the data in the equation (8).

The control limits for $\bar{x}$ chart are

$$\begin{cases} UCL = \bar{\bar{x}}_w + 3\dfrac{1}{\sqrt{n}\,c_n}\bar{s}_w \\ CL = \bar{\bar{x}}_w \\ LCL = \bar{\bar{x}}_w - 3\dfrac{1}{\sqrt{n}\,c_n}\bar{s}_w \end{cases} \quad (13)$$

The control limits for s chart are

$$\begin{cases} UCL = \bar{s}_w\left(1+3\dfrac{1}{c_n}\sqrt{\dfrac{n-1}{n}-c_n^2}\right) \\ CL = \bar{s}_w \\ LCL = \max\left\{0, \bar{s}_w\left(1-3\dfrac{1}{c_n}\sqrt{\dfrac{n-1}{n}-c_n^2}\right)\right\} \end{cases} \quad (14)$$

Here

$$c_n = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\sqrt{\frac{2}{n}}$$

3. Chloride content data example

The present data contains the observations of the chloride content in soda ash given the upper specification limit (USL) 1 per cent measured at the fixed times 0, 4, 8, 12, 16 and 20 hrs in a day for 16 days. There are 16 subgroups or samples each of size 6 in the data. The subgroups are rationally taken.

Our aim is to construct robust $\bar{x}$ and the s charts. We plot the usual subgroup statistics sample mean and sample standard deviation, but we use all the three methods to compute the control limits and lastly we make a comparison of first two methods with the last the proposed one.

The computed values of control limits computed using Shewhart, MED-MAD, and WLE methods are tabulated in the tables 2, 3, and 4. Looking at the tabulated values of the control limits, one easily get the idea that the control limit calculated by using the WLEs are more appropriate and robust than the usual method of Shewhart charts. Moreover the WLE control limits are close to those calculated by using MED and MAD estimates.

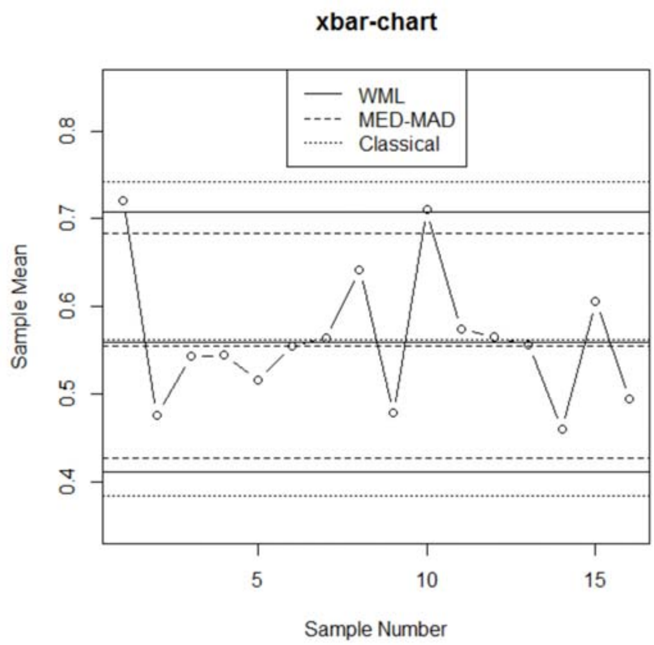


Fig. 1 : Chloride content data: $\overline{X}$ Chart

To illustrate the proposed method of constructing the control limits for $\bar{x}$ and *s* charts, we give the control chart plots with control limits established by classical method (dotted lines), by MED-MAD method (dashed lines) and by WLE method based on Method 1 with weight function $W_1(0.1)$ (solid lines) in the figures 1 and 2.

Now we give the estimates of process capability index, process fallout using these all three estimates of process parameters given *USL* = 1%.

Let us define measures of performance and capability of the process, which is useful given *USL* = 1%.

The upper process capability index in per cent

$$C_{pku} = \frac{USL - \mu}{3\sigma} \times 100$$

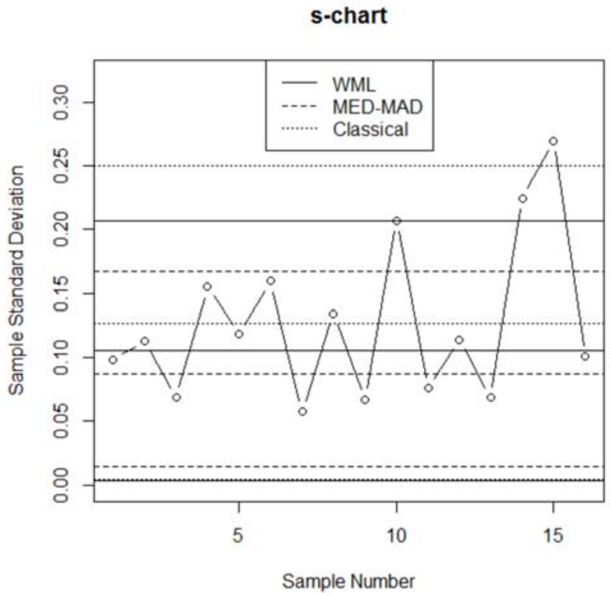


Fig. 2 : Chloride content data: *S* Chart

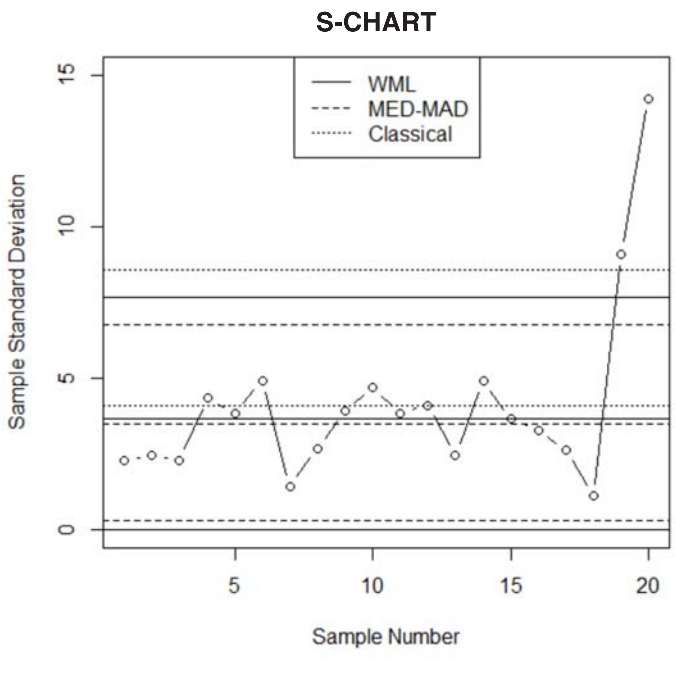


Fig. 3 : Textile Data: X Chart

The percentage upper specification band used by the process as

$$P_u = \frac{1}{C_{pku}} \times 100$$

Normalized USL

$$Z_u = \frac{USL - \mu}{\sigma}$$

Let random variable Z ~ N (0, 1), the upper ppm (part per million) defective or upper process fallout

$N_u = Pr\,[Z > Zu] \times 10^6$;

To estimate the measures of performance and capability of the process using the control charts based on the different methods, we uses the unbiased estimates of the process mean and standard deviation under the respective method.

Table 5 : The estimated values of the measures of capability and performance of the process using the parameter estimates based on usual method, on MED-MAD method and on the WLE method 1 from chloride content data

	Classical Method	MED-MAD	WLE Method 1		
			$w_1(c = 0.1)$	$w_2(c = 0.3)$	$w_3(c = 0.3)$
C_{pku}	0099.709	0142.013	0121.405	0121.346	0121.369
P_u	0100.291	0070.416	0082.368	0082.408	0082.393
Z_u	0002.991	0004.260	0003.642	003.640	0003.641
N_u	1389.041	0010.203	0135.171	0136.107	0135.749

One can observed in the table 5 that estimated values of the performance and capability of the process based on the WLE are optimally better than those calculated by usual statistics. The estimated values of the performance measures of the process based on MED and MAD are highly inflated because these estimates lack in efficiency when compared with the WLE.

Hence, we recommend these control limits based on the WLE method to perform statistical process control on this process in future. One can alternatively use these control limits to homogenize and revise the control limits based on Shewhart method. Moreover the WLE method can be used to quickly and rapidly find out the very data point present in any subgroup sample in the data and is responsible for pulling this subgroup out of the control limits. This may help the management to find out the exact reason of outlying subgroup.

4. Textile Data Example

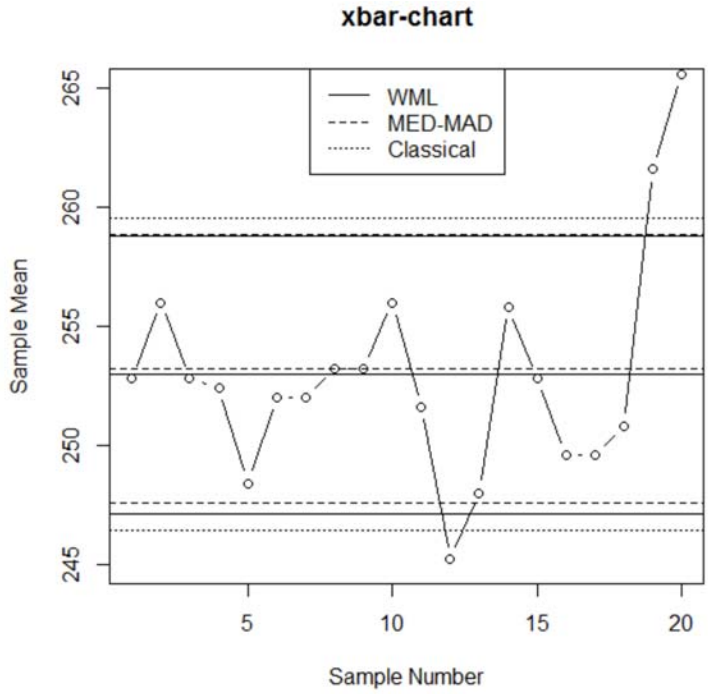
This data contains measurements on the linear density of cotton sliver. It is taken from Damyanov and Germanova-Krasteva (2013) Five measurements (n = 5) have been made in a period of 20 days (m = 20) and the linear density has been recorded in Number metric (Nm). The tabulated values have been multiplied by 1000 *i.e.* when the measured value has been Nm 0.345 then the value recorded in the table is 345.

To illustrate the proposed method of constructing the control limits for $\bar{x}$ and *s* charts, we give the control chart plots with control limits established by classical method (dotted lines), by MED-MAD method (dashed lines) and by WLE method based on Method 2 with weight function $W_1(0.0001)$ (solid lines) and tail probability $p = 0.5$ in the figures 3 and 4.

Looking at the results in the table 6 and the figures 3 and 4, one can easily find control limits based on the WLE method more reliable and efficient to perform statistical process control on this process in future. Alternatively it can be used to homogenize and revise control limits based on Shewhart method and moreover this proposed method can be used to detect the very data point present in any subgroup sample in the data and is responsible for pulling this subgroup out of the control limits, which in turn may help the management to find out the exact reason of outlying subgroup.

5. Simulation Study

In this section, we try to establish our claim that the proposed method is optimally give better estimates for location and scale parameters of the process than any other existing procedure and hence the constructed control limits are rapid and accurate in detecting out of control situations if any.



Procedure

Step 1. Generate n observations from a (normal) population or mixture of (normal) populations (i.e. contaminated normal population) with parameters say μ and σ^2.

Step 2. Calculate the different location and scale estimates that are sample mean, sample median, sample weighted mean, sample standard deviation, sample median absolute deviation from median and sample weighted standard deviation.

Step 2. Repeat Step 1 & 2 m times.

Step 3. Find the average of each of the statistics computed in Step 2.

Step 4. Repeat Step 1, 2 and 3 1000 times.

Step 5. Calculate the mean square errors for each of the estimates.

We have done this simulation for different combinations of m and n, the simulation results are tabulated below. Looking at the tables it is clear that in the most of the situations the proposed estimates have lesser mean square error as compared to peer estimators. Hence one can conclude that the proposed method of constructing control limits are better and competitive with the other methods and could rapidly detects any erroneous points in the data.

Table 6 : The control limits for textile data computed by Shewhart, MED-MAD, and WLE methods. The method 2 with all the three weight functions used to calculate WLEs, where tail probability p = 0.5

	Classical Method	MED-MAD	WLE		
			$w_1(c = 0.001)$	$w_2(c = 0.015)$	$w_3(c = 0.015)$
Control limits for x chart					
LCL	246.423825	247.562645	247.128327	247.142029	247.142835
CL	252.970000	253.200000	252.974493	252.977160	252.977482
UCL	259.516175	258.837355	258.820659	258.812292	258.812128
Control limits for s chart					
LCL	0	0.298252	0	0	0
CL	4.102207	3.484110	3.663541	3.656626	3.656322
UCL	8.569501	6.767125	7.653130	7.638685	7.638050

Table 7 : Mean square errors of different location and scale estimates of the process for WLE method 1 with weight function W_1(0.01) for n = 5, m = 20

Parameter	Population	Classical	MED-MAD Method	WLE
μ	N(0, 1)	00.0100	0.0139	0.0113
	0.9N(0,1)+0.1N(25,1)	06.7355	0.3707	0.2991
	0.9N(0,1)+0.1N(0,25)	00.0341	0.0185	0.0172
	0.8N(0,1)+0.2N(25,1)	25.7342	4.4658	3.8620
	0.8N(0,1)+0.2N(0,25)	00.0614	0.0269	0.0278
σ	N(0,1)	00.0095	0.0420	0.0556
	0.9N(0,1)+0.1N(25,1)	20.2710	0.0376	0.0606
	0.9N(0,1)+0.1N(0,25)	00.2810	0.0214	0.0335
	0.8N(0,1)+0.2N(25,1)	56.9408	0.2266	0.0656
	0.8N(0,1)+0.2N(0,25)	01.0448	0.0749	0.0191

One can noticed the simulation results in the tables 7, 8, 9 and 10, that as the subgroup sample size n increases, the MSEs for WLE become smaller as compared to the that of MAD-MED estimators. That is for large n, the WLEs behave like MLEs if the data are pure, and act efficiently and robustly better than MLEs under contamination.

Table 8 : Mean square errors of different location and scale estimates of the process for WLE method 1 with weight functionW_1(0.01) for n = 10, m = 20

Parameter	Population	Classical	MED-MAD Method	WLE
μ	N(0,1)	00.00493	0.00675	0.00506
	0.9N(0,1)+0.1N(25,1)	06.63679	0.05657	0.03796
	0.9N(0,1)+0.1N(0,25)	00.01797	0.00873	0.00725
	0.8N(0,1)+0.2N(25,1)	25.93464	1.07966	0.83351
	0.8N(0,1)+0.2N(0,25)	00.03028	0.01063	0.01157
σ	N(0, 1)	00.00350	0.01336	0.01246
	0.9N(0,1)+0.1N(25,1)	29.49744	0.06143	0.01413
	0.9N(0,1)+0.1N(0,25)	00.40360	0.00992	0.00599
	0.8N(0,1)+0.2N(25,1)	72.40148	1.18724	0.01620
	0.8N(0,1)+0.2N(0,25)	01.36438	0.05136	0.01758

Table 9 : Mean square errors of different location and scale estimates of the process for WLE method 1 with weight functionW1(0:01) for n = 5, m = 40

Parameter	Population	Classical Method	MED-MAD	WLE
μ	N(0, 1)	00.00514	0.00699	0.00601
	0.9N(0,1)+0.1N(25,1)	06.52806	0.25291	0.17286
	0.9N(0,1)+0.1N(0,25)	00.01722	0.00886	0.00878
	0.8N(0,1)+0.2N(25,1)	25.74433	3.70936	2.99403
	0.8N(0,1)+0.2N(0,25)	00.02689	0.01282	0.01310
σ	N(0, 1)	00.00643	0.03617	0.05330
	0.9N(0,1)+0.1N(25,1)	19.5971	0.02281	0.05675
	0.9N(0,1)+0.1N(0,25)	00.25547	0.01131	0.03071
	0.8N(0,1)+0.2N(25,1)	57.44086	0.20651	0.06165
	0.8N(0,1)+0.2N(0,25)	00.99263	0.05895	0.01176

Table 10 : Mean square errors of different location and scale estimates of the process for WLE method 1 with weight function $W_1(0.01)$ for n = 10, m = 40

Parameter	Population	Classical Method	MED-MAD	WLE
μ	N(0, 1)	00.00235	0.00305	0.00242
	0.9N(0;1)+0.1N(25,1)	06.47093	0.04562	0.01834
	0.9N(0;1)+0.1N(0,25)	00.00836	0.00404	0.00361
	0.8N(0;1)+0.2N(25,1)	25.21211	0.84980	0.57789
	0.8N(0;1)+0.2N(0,25)	00.01447	0.00516	0.00537
σ	N(0,1)	00.00202	0.01012	0.01106
	0.9N(0,1)+0.1N(25,1)	29.11023	0.03712	0.01217
	0.9N(0,1)+0.1N(0,25)	00.39618	0.00557	0.00307
	0.8N(0,1)+0.2N(25,1)	71.90268	0.94660	0.01315
	0.8N(0,1)+0.2N(0,25)	01.36868	0.04775	0.01329

CONCLUSION :

In this paper, we have proposed a robust alternative for constructing the control limits for $\overline{X}$ and *S* control charts based on a minimum distance approach. This proposal aims at building resistant control limits for the control charts; the subgroup statistics which are to be plotted on the charts should be sensitive to outliers, so that any slight shift in the process could be easily and rapidly detected. To give a clear illustrations of the proposed method, two real data examples and a simulation study has been given. It is observed that the control charts established by the proposed technique can ameliorate the troubles engendered by presence of the possible outliers in the data and thus perform optimally or competitively better than the available techniques.

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