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ON VARIANCE BALANCE AND EFFICIENCY BALANCE IN BLOCK DESIGNS

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ABSTRACT

From considerations of analytical simplicity of block designs we characterize the two important concepts of balance, namely, variance balance and efficiency balance, and discuss some properties and constructions of efficiency balanced designs. We also discuss another interesting characterization of variance and efficiency balance of block designs.

1. Introduction

The objective of the article is twofold: (i) to discuss two characterizations of variance balance and efficiency balance in block designs, specically to show (a) how 'simplicity' of analysis and 'balance' of a block design are inter-related and (b) how a clear-cut and sounder definition of eficiency balance, much different from what is found in early literature, can be offered; (ii) to discuss certain combinatorial properties and constructions of efficiency balanced designs.

Consider a block design with the parameters (v, b, r, k), where, v is the number of treatments, b is the number of blocks, $r = (r_1, r_2, \ldots ; r_v)'$ is the vector of replication numbers of the treatments, and $k = (k_1, k_2, \ldots , k_b)'$ is the vector of block sizes. Denote the incidence matrix of the block design by $N = ((n_{ij}))_{v\times b}$ and let T be the vector of treatment totals, and B be that of block totals. Then the reduced normal equations for the treatment effects $\underline{\tau}$ are given by

$$C\hat{\underline{\tau}} = \underline{Q},$$

where, $C = \underline{r}^{\delta} - Nk^{-\delta}N'$, $\quad \underline{Q} = \underline{T} - Nk^{-\delta}\underline{B}$,

$$r^{\delta} = \text{Diag}\,[r_1, r_2, \ldots, r\ ]$$

$$k^{-\delta} = (k^{\delta})^{-1} = \text{Diag}\,[k_1\ , k_2^{-1}, \ldots, k_b^{-1}],$$

and N' is the transpose of N. The central problem of the analysis of a block design is, thus, to get hold of a g-inverse, say C^-, of C, that is, a matrix C^- satisfying $C C^- C = C$. Thus the C-matrix of a block design plays a very important role in its analysis.

Calinski (1971) and some other authors have given more importance to what is called the M_0-matrix of a block design. In fact the very concept of efficiency balance in most of the literature is intimately associated with this M_0-matrix. We shall, however, show in this article that we can forget about M_0 and "balance" of Jones [1959] and his unusual assumptions to interpret the "loss of information" (μ), and still talk about the eficiency balance of a block design on the basis of only the C-matrix. Incidentally, throughout this paper, we shall talk about connected designs only, for which *rank* (C) = v -1.

Tocher [1952] showed that $C + \underline{r}\,\underline{r}'/n$ is non-singular, where $n = \underline{1}'\underline{r} = \underline{1}'\underline{k}$, $\underline{1}'$ being the row vector of all 1's of appropriate size, and that the true inverse of $C + \underline{r}\,\underline{r}'/n$ is a g-inverse of C. Thus

$$\Omega^{-1} = C + \underline{r}\underline{r}'/n, \text{ where } C\,\Omega\,C = C.$$

Now

$$\Omega^{-1} = r^{\delta}[I - Mo], \text{ where } Mo = r^{-\delta}Nk^{-\delta}\,\underline{1}\underline{r}'/n.$$

Calinski [1971] has shown that

$$\Omega = [I_v - M_0]^{-1}r^{-\delta} = \left[I_v + \sum_{h=1}^{\infty} M_0^h\right]r^{-\delta}$$

which reduces to a very simple form for, what he called, totally balanced designs, or, efficiency balanced designs (EBD). In fact, from the following current denition of EBD, Calinski [1971] observes that a block design is an EBD if and only if

$$M_0 = \mu[I_v - \underline{1}\underline{r}'/n,], 0 < \mu < 1.$$

A block design is efficiency balanced if and only if, for every treatment contrast vector s, (Note that this is an observational contrast so that $\underline{s}'\underline{r} = 0$.)

$$M_0\underline{s} = \mu\underline{s}, \textit{for some real } \mu,\ 0 < \mu < 1,$$

where μ is the unique non-zero eigenvalue of M_0.

It is easy to see that for an EBD,

$$\Omega = [(1-\mu)^{-1} I_v + \mu(1-\mu)^{-1} \underline{1}\,\underline{r}' / n] r^{-\delta}$$

and hence for every (parametric) contrast $\lambda'\tau$,

$$V(\lambda'\underline{\hat{\tau}}) = \lambda'\Omega\lambda\sigma^2 = (1-\mu)^{-1} \underline{\lambda}' r^{-\delta} \lambda\sigma^2$$

Therefore, the relative information gained on $\underline{\lambda}'\underline{\tau}$, (compared to an orthogonal design), or efficiency on $\underline{\lambda}'\underline{\tau}$, is

$$\lambda' r^{-\delta} \lambda\sigma^2 (1-\mu)^{-1} \lambda' r^{-\delta} \lambda\sigma^2 = (1-\mu).$$

Hence the name "efficiency balanced" designs for the above class of block design.

We end this section by recalling the definition of variance balance which considers only the parametric contrasts and not observational contrasts.

A block design is variance balanced (VB) if and only if V ($\lambda'\underline{\hat{\tau}}$)/ $\lambda'\lambda$ is a constant for all (parametric) contrast vactors λ :

2. Two Characterizations

2.1 The results of this sub-section are from Sinha and Saha [1985]. Although they are true for a general block design, we shall consider only the connected design version, as pointed out in the previous section.

Parallel to and consistent with Definition 2, we define an efficiency balanced design as follows. And we shall use this denition in all the results of this section.

A block design is efficiency balanced if and only if $\underline{\lambda}' r^{-\delta} \lambda / V$ ($\underline{\lambda}'\underline{\hat{\tau}}$) is a constant for all (parametric) contrast vectors λ.

(Note: For $\underline{r} = r \cdot \underline{1}$, Denition 1 and Denition 2 are equivalent.)

We shall now see how the analytical simplicity of a block design and its variance balance or efficiency balance are inter-related. If C^- of a block design can be obtained easily and immediately without any computer assistance, then the design can be called a "simple" (to analyze) design. While commenting on "simplicity" of a block design in an age of computers, Calinski [1971] remarks that 'simplicity' is usually accompanied with elements of "balance" of the design. It is clear that balanced designs are "simple." We ask: are "simple" designs balanced? The answer is yes. For example,

(i) $C^- \propto I_v \Leftrightarrow$ variance balance (Definition 1);

(ii) $C^- \propto r^{-\delta} \Leftrightarrow$ eficiency balance (Definition 2).

Thus, the analytical simplicity of certain types characterize the variance balance and efficiency balance of a block design. From this characterization it is now easy to characterize the *C*-matrix of a VB and EB design and arrive at the results of Rao [1958], Kageyama [1974], Calinski [1971] *etc*., on such designs. In fact, it can be proved that

(i) V ($\underline{\lambda}'\underline{\hat{\tau}}$)/ $\underline{\lambda}'\underline{\lambda}$ constant for all $\underline{\lambda}$ such that $\underline{\lambda}'$ 1 = 0

if and only if $C^- \propto I_v$

if and only if $C^- \propto [I_v - v^{-1} Jvv]$,

where J_{uu} is a $v \times v$ matrix of all ones;

(ii) $\lambda' r^{-\delta} \lambda / \lambda' C^- \lambda$ =constant for all $\underline{\lambda}$ such that $\underline{\lambda}'\underline{1}$ =0

if and only if $C^- \propto r^{-\delta}$

if and only if C [$r^\delta - \underline{r}\,\underline{r}' / n$]

if and only if $M_0 \propto [I_v - \underline{1} r' / n]$.

Thus we obtain here a simple and straightforward characterization of efficiency balanced design based on a neat and clear-cut definition of EBD (Def.2).

2.2 Jones [1959] considered a diffierent characterization of what is today called EBD. Instead of considering parametric contrasts, he considers observational contrasts and their intra- and inter- block components. $\underline{s}'\underline{T}$, where $\underline{s}'\underline{r} = 0$, is called a **treatment contrast**.

Now,

$$\underline{s}'\underline{T} = \underline{s}'\underline{Q} + \underline{s}' N \underline{k}^{-\delta} \underline{B}$$

$= intra - blockcomp. + inter -$ blockcomp.,

where

$$Cov(\underline{s}'\underline{Q}, \underline{s}' N k^{-\delta} \underline{B}) = 0$$

Thus the two components are mutually orthogonal observational contrasts. Let $\underline{s_1}$ and $\underline{s_2}$ be two mutually orthogonal treatment contrast vectors, that is,

$$\underline{s_1}'\underline{r} = \underline{s_2}' r = 0 = \underline{s_1}' r^\delta \underline{s_2}.$$

When will Cov($\underline{s_1}'\underline{Q}, \underline{s_2}'Q$) be zero? Jones [1959] came up with the following answer:

$$Cov(s_{\underline{1}}'\underline{Q}, s_{\underline{2}}'Q) = 0 \Leftrightarrow \sum_{u=1}^{b} n_{iu} n_{ju} / k_u \propto r_i r_j$$

for all i, j; i ≠ j = 1; 2; ...; v:

The right hand side is again equivalent to (See Puri & Nigram [1977].)

$$C = (1-\mu)[r^{\delta} - \underline{r}\,\underline{r}'/n],$$

where μ is some real constant, $0 < \mu < 1$. Thus the intra-block components of orthogonal treatment contrasts are orthogonal for EBD, and only EBD.

A similar characterization for variance balanced design can be obtained. One can really establish that:

$$Cov\,(\underline{\lambda}'\hat{\underline{\tau}}, \underline{m}'\hat{\underline{\tau}}) = 0, \text{ for all } \underline{\lambda}, \underline{m} \text{ such that } \underline{\lambda}'\underline{m} = 0$$

$$= \underline{\lambda}'1 = \underline{m}'1$$

$$\Leftrightarrow C^{-} \propto Iv$$

$\Leftrightarrow$ *the design is variance balanced.*

Thus BLUE's of orthogonal treatment contrasts are also orthogonal if and only if the design is variance balanced. The result is true for disconnected designs as well. Below we given an outline of the proof for the general case.

Since $Cov\,(\lambda'\hat{\underline{\tau}}, \underline{m}'\hat{\underline{\tau}}) = \lambda' C^{-}\underline{m}\sigma^2 = \underline{q}'CC^{-}C\underline{p}\sigma^2$ for some $\underline{p}$ and $\underline{q}$, the given condition of zero covariances is equivalent to $\underline{q}'C\underline{p} = 0$, for all p, q such that $\underline{q}'C^2\underline{p} = 0'$ Again this is equivalent to $C \propto C^2$, i.e., $C = C(\theta I_v)C$ for some constant θ. Hence the result.

3. Some properties of EB designs

Kageyama [1980] has shown several interesting properties of efficiency balanced designs. They can be classified into the following four categories:

(i) inequalities (or upper and lower bounds, as they are popularly called) involving the constants, v, b, r_i, k_j and μ;

(ii) necessary conditions for the existence of EB designs;

(iii) results on the number of disjoint blocks of symmetric ($v = b$) binary EB designs;

(iv) results on dual of an EB design.

For details a reference may be made to the lengthy paper of Kageyama [1980].

Saha [1976] established that the dual of an EB design is a C-design, *i.e.*, a block design satisfying $M_0^2 = \mu M_0$, $0 < \mu < 1$. Kageyama [1980] also made the same observation.

Puri and Nigam [1975] had shown that if $N_{v\times b}$ is the incidence matrix of an EB design, then so is $N^*_{t\times b}$ where N^* is obtained from N by adding the first a_1 rows, then next a_2 rows, ..., up to the last a_t rows of N, where $a_1+a_2+ +a_t=v$.

We close this section with some observations on the bounds of Kageyama [1980] on r_i and μ of an EB design. He has shown that for a binary EB design,

$$n - (\min_i nr_i)(\max_j xk_j)(\max_j xk_j)(n - \min_i nr_i)$$

$$\le \mu \le n - (\max_i xr_i)(\min_j nk_j)(\min_j nk_j)(n - \max_i xr_i) \quad (3.1)$$

and

$$n1 - \mu(1\max_j xk_j - \mu) \le r_i \le n1 - \mu(1\min_j nk_j - \mu) \quad (3.2)$$

Now from the C-matrix of an EB design, given by

$$C = (1-\mu)[r^{\delta} - \underline{r}\,\underline{r}'/n], \quad (3.3)$$

we have, on comparing the diagonal elements of both the sides,

$$\mu = [\sum_{j=1}^{b} n_{ij}^2 / = k_j - r_i^2/n]/[r_i = r_i^2/n], \quad (3.4)$$

from which we can write

$$\mu = [n\sum_i\sum_j n_{ij}^2/k_j - a]/[n^2 - a],\ a = \sum_i r_i^2 \quad (3.5)$$

Clearly then, if the design is binary, *i.e.*, if n_{ij}=0 or 1,

$$\mu = [nb - a]/[n^2 - a], \quad (3.6)$$

as observed by Kageyama [1980], also. When an exact relation is known an inequality ceases to have a meaning. It is for this reason that (3.1) is useless in view of (3.6).

Again, from (3.4) we can find an upper bound for ri, not involving μ and hence quick-to-apply, which may be preferable to that of (3.2). Since $0 < \mu < 1$, we have from (3.4):

$$r_i^2 n < \sum_j n_{ij}^2 k_j < r_i, i = 1, 2, ..., v. \quad (3.7)$$

Therefore, for binary EB design

$$r_i^2 n < \sum_j n_{ij} k_j \le \sum_j 1k_j,$$

and hence $$r_i \leq \sqrt{n\sum_j \frac{1}{k_j}} \qquad (3.8)$$

Utilizing the other half of the inequality (3.7), namely

$$r_i > \sum_{j=1}^{b} n_{ij}^2 / k_j$$

which incidentally is true not only for an EB design, but also for a general block design, we can write

$$\sum_i r_i > \sum_i \sum_j n_{ij}^2 k_j$$

that is,

$$n > \sum_i \sum_j n_{ij}^2 k_j \geq (\sum_i \sum_j n_{ij} k_j)^2 / (\sum_i \sum_j 1k_j). \qquad (3.9)$$

From this inequalitiy, it follows that

$$n > b^2 / (v\sum_j 1k_j), or \sum_j 1k_j > b^2 nv \qquad (3.10)$$

an inequality involving the parameters of a general block design.

Finally, we give a lower bound for μ for any EB design (not assumed to be binary only). From (3.5), (3.9) and (3.10) it follows

$$\mu \geq (nb^2(v\sum_j 1k_j) - a)n^2 - a \qquad (3.11)$$

We end with a note on upper bounds of b in EB design. A well-known lower bound for the number of blocks of EB design is

$$b \geq v \qquad (3.12)$$

(Kageyama [1980]. From (3.10), we have an upper bound for b :

$$b \leq \sqrt{nv\sum_j 1k_j}. \qquad (3.13)$$

Again substituting for μ from (3.6) in μ of (3.11), we have for binary EB design

$$b \leq v\sum_j 1k_j \qquad (3.14)$$

4. Constructions of EB designs

There is really no theory for the construction of efficiency balanced designs. In literature there exist some discrete methods of construction of such designs.

Saha [1976] constructed an EB design by adding a new treatment suitably to a BIBD along with its complementary design with parameters satisfying a certain relation. Kulshreshtha *et al*. [1972], and Hedayat and Federer [1974] gave a method, called "method of unionizing" of construction of variance balanced designs. Later, Puri and Nigam [1977] generalized this method for the construction of efficiency balanced designs.

Recently, Kageyama [1981] considered the construction of binary EB designs. Basically, he generalized Puri and Nigam's [1977] method in various directions and constructed a large number of binary EB designs. For details of the interesting constructional results a reference may be made to the paper of Kageyama [1981].

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