ESTIMATION OF THE SMALLEST LOCATION OF TWO NEGATIVE EXPONENTIAL POPULATIONS

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ABSTRACT

We propose an estimator of the smallest location of two negative exponential populations. The proposed estimator is compared with the existing maximum likelihood estimator through different performance criteria. In support of these comparisons, we also provide some numerical computations.

Keywords *:* Exponential distribution; Location and scale parameter; Maximum likelihood estimator; Bias; Mean square error; Pitman Closeness

1. Introduction

Estimation of the smallest (largest) location of several exponential populations has always been an interesting problem. The problem of estimating the largest mean of K normal populations was considered by Kuo and Mukhopadhyay (1990), Mukhopadhyay *et al* (1993), Saxena and Tong (1969) and Tong (1970), among others. A related problem of estimating the largest component of a multinomial parameter has recently been considered by Alam and Feng (1997). Kuo and Mukhopadhyay (1990a) considered the point estimation problem of the largest location of K negative exponential populations. But, most of these works are for fixed-width interval estimation based on sequential or multi-stage sampling.

In this paper, we consider the point estimation of the smallest location of two negative exponential populations in the face of unknown scale parameter(s). In the language of reliability and life testing, this amounts to the estimation of the minimum guarantee time when the assumed distributions are all exponentials with unknown failure rate(s). The exponential distribution has been widely used for describing the distribution of failure times of complex equipment, vacuum tubes and other small components. Applications of simple exponential models are concerned with animal tumour systems and acute leukaemia, destruction of tumour cells with laser energy, and the analysis of survival data with concomitant information. One can see Zelen (1996) in this context.

Let $E(\mu_i, \sigma_i), i=1,2$. be two independent negative exponential populations and $(X_{i1},..., X_{ini})$ be a sample from the i[th] population. Our objective is to estimate $\theta=\mu_{(1)}=min(\mu_1, \mu_2)$. First, we consider the case of equal scale parameter, *i.e.* $\sigma_1=\sigma_2=\sigma$, say. We propose an estimator of θ with a slight modification over the existing maximum likelihood (ML) estimator in section 3. The suggested estimator may be useful to estimate the minimum guarantee time of a series system comprising of two components having independent exponential life distributions. In section 4, the proposed estimator is compared with the ML estimator both asymptotically, and also in terms of their small sample performance. Next, in section 5, we consider the case of unequal scale parameters. Some numerical computations for the case of equal scale parameter are also provided to justify our findings.

2. Basic Distributions

Since life-time is non-negative, it is quite reasonable to assume $\mu_1>0$, $i=1,2$. Also $\sigma>0$, $X_{ij}>\mu_i \forall j=1(1)n_i i=1,2$, Let

$$\left.\begin{aligned} &Y_i = min\{X_{ij}, J=1(1)n_i\}, \quad \ldots (2.1)\\ &S_i=\textstyle\sum_{i=1}^{ni}(X_{ij}-Y_i),\ i=1,2\\ &\text{and } Y = min(Y_1, Y_2). \end{aligned}\right\}$$

It is well-known that

$$\left.\begin{aligned} &Y_i \sim E\left(\mu_i, \frac{\sigma}{n_i}\right), Y_i > \mu_i, i=1,2 \quad \ldots (2.2)\\ &\frac{2S_i}{\sigma} \sim X^2_{2(ni-1)}, \end{aligned}\right\}$$

and the variables are all independent.

Now the distribution function (d.f.) of Y_i is given by

$$F_i(y) = 1-exp\left[-\frac{n_i(y-\mu i)}{\sigma}\right] \text{ if } y > \mu_i$$
$$0 \quad \text{if } y < \mu_i, i=1,2 \quad \ldots (2.3)$$

If $G(y)$ is the d.f. of Y then, using (2.3), we get

$$1-G(y) = \prod_{i=1}^{2} P(Y_i > y)$$

$$= \begin{cases} 1 & \text{if } y \le \mu_{(1)} \\ exp\left[-\frac{n^{(1)}(y-\mu_{(1)})}{\sigma}\right] & \text{if } \mu_{(1)} < y \le \mu_{(2)} \\ exp\left[-\frac{1}{\sigma}\sum_{i=1}^{2} n_i(y-\mu_i)\right] & \text{if } y > \mu_{(2)} \end{cases} \quad \ldots(2.4)$$

where

$\mu_{(1)} = \min(\mu_1, \mu_2) \le \mu_{(2)} = \max(\mu_1, \mu_2)$

and $n^{(i)}$ is the sample size corresponding to the population having location $\mu_{(i)}, i=1,2$. Clearly

$$n^{(i)} = n_i \text{ when } \mu_{(i)} = \mu_i,\ i=1,2$$

Let

$$N = n_1 + n_2 = n^{(1)} + n^{(2)}$$

and

$$\bar{\mu} = \frac{1}{N}\sum_{i=1}^{2} n_i \mu_i = \frac{1}{N}\sum_{i=1}^{2} n^{(i)} \mu_{(i)}$$

$\mu_{(1)}$ is usually estimated by

$$T = min(\hat{\mu}_1, \hat{\mu}_2) \quad \ldots \ (2.5)$$

where $\hat{\mu}_i$ is an estimator of μ_i. If $\hat{\mu}_i = Y_i$ the maximum likelihood (ML) estimator of μ_i, we get $T = \min(Y_1, Y_2) = Y$, which is the ML estimator of $\mu_{(1)}$. Let us denote this estimator by T_M.

3. Proposed Estimator

In this section we propose an estimator which reduces the bias in T_M. The bias and mean square error (MSE) of T_M can be easily obtained as

$$Bias\ (T_M) = \frac{\sigma}{n^{(1)}}(1-e^{-\mu^*}) + \frac{\sigma}{N}e^{-\mu^*} \quad \ldots \ (3.1)$$

$$MSE(T_M) = \frac{2\sigma^2}{n^{(1)^2}} + e^{-\mu^*}\Big[\mu^{-2} - \mu_{(1)}^2 + \frac{2\sigma}{N}\bar{\mu}(\mu^*+1)$$

$$+\sigma^2\left(\frac{1}{N^2} - \frac{1}{n^{(1)^2}}\right)\left\{\left(\mu^*+1\right)^2+1\right\}$$

$$-2\sigma\mu_{(1)}\left(\frac{\mu^+}{n^{(1)}} + \frac{1}{N}\right)\Big] \quad \ldots(3.2)$$

where

$$\mu^* = \frac{n^{(1)}(\mu_{(2)} - \mu_{(1)})}{\sigma} = \frac{N(\mu_{(2)} - \bar{\mu})}{\sigma} \quad \ldots \ (3.3)$$

We see that $Bias(T_M) > 0,$ and hence T_M overestimates $\mu_{(1)}$. To reduce the bias, we propose the following modified estimator

$$T^* = Y - \frac{\hat{\sigma}}{\hat{n}^{(1)}} \quad \ldots \ (3.4)$$

where

$$\hat{\sigma} = \frac{1}{N-2}(S_1 + S_2$$

and

$$\hat{n}^{(1)} = n_i \text{ if } Y = Y_i, i = 1, 2.$$

Since S_i and Y_i are independent, $\hat{\sigma}$ and $\hat{n}^{(1)}$ are also independent. Hence

$$E(T^*) = E(Y) - E(\hat{\sigma})E\left(\frac{1}{\hat{n}(1)}\right) \quad \ldots \ (3.5)$$

Here

$$E(\hat{\sigma}) = \sigma, E\left(\frac{1}{\hat{n}^{(1)}}\right) = \sum_{i=1}^{2}\frac{1}{n^i}P(Y = Y_i),$$

where

$$P(Y = Y_1) = 1 - P(Y = Y_2) = \begin{cases} 1 - \frac{n^{(2)}}{N}e^{-\mu^*} \\ \frac{n^{(2)}}{N}e^{-\mu^*} \end{cases} \text{if} \begin{cases} \mu_{(1)} = \mu_1 \\ \mu_{(1)} = \mu_2 \end{cases} \ldots (3.6)$$

Then it can be shown that

$$E\left(\frac{1}{n^{(1)}}\right) = \frac{1}{n^{(1)}} + \frac{1}{N}e^{-\mu^*}(1-\rho) \quad \ldots \ (3.7)$$

where

$$\rho = \frac{n^{(2)}}{n^{(1)}}$$

Thus,

$$E(T^*) = \mu_{(1)} + \frac{\sigma}{n^{(1)}} + \sigma e^{-\mu^*}\left(\frac{1}{N} - \frac{1}{n^{(1)}}\right) - \sigma\left[\frac{1}{n^{(1)}} + \frac{1}{N}e^{-\mu^*}(1-\rho)\right]$$

$$= \mu_{(1)} - \frac{\sigma}{N}e^{-\mu^*} \quad \ldots \ (3.8)$$

implying that

$$Bias\left(T^*\right) = -\frac{\sigma}{N} e^{-\mu^*} < 0, \qquad \ldots \quad (3.9)$$

and thus T^* underestimates $\mu_{(1)}$

Now to find the MSE of T^*, we see that

$$V\left(Y - \frac{\hat{\sigma}}{\hat{n}^{(1)}}\right) = E\left(Y - \frac{\hat{\sigma}}{\hat{n}^{(1)}}\right)^2 - E^2\left(T^*\right)$$

$$= E\left(Y^2\right) + E\left(\hat{\sigma}^2\right) E\left(\frac{1}{n^{(1)^2}}\right) - 2\sigma E\left(\frac{Y}{n^{(1)}}\right) - E^2\left(T^*\right). \qquad \ldots \quad (3.10)$$

$E\,(Y^2)$ and $E\left(\hat{\sigma}^2\right)$ can be easily obtained by using the distribution of Y and $\hat{\sigma}$. Also, after simple manipulations, we get

$$E\left(\frac{1}{n^{(1)^2}}\right) = \sum_{i=1}^{2} \frac{1}{n_i^2} P\left(Y = Y_i\right)$$

$$= \frac{1}{n^{(1)^2}} + \frac{n^{(2)}}{N} e^{-\mu^*} \left(\frac{1}{n^{(2)^2}} - \frac{1}{n^{(1)^2}}\right) \qquad \ldots \quad (3.11)$$

and

$$E\left(\frac{Y}{n^{(1)}}\right) = EE\left(\frac{Y}{n^{(1)}} | Y\right)$$

$$= \sum_{i=1}^{2} E\left(\frac{Y_i}{n_i}\right) P\left(Y = Y_i\right)$$

$$= \left(\mu_{(2)} + \frac{\sigma}{n^{(2)}}\right) \frac{e^{-\mu^*}}{N} + \left(\mu_{(1)} + \frac{\sigma}{n^{(1)}}\right) \frac{1}{n^{(1)}} \left(1 - \frac{n^{(2)}}{N} e^{-\mu^*}\right) \ldots (3.12)$$

Thus

$$MSE\left(T^*\right) = \frac{2\sigma^2}{n^{(1)^2}}$$

$$+ e^{-\mu^*} \left[\mu^2 - \mu_{(1)}^2 + \sigma^2 \left(\frac{1}{N^2} - \frac{1}{n^{(1)^2}}\right) \left\{\left(\mu^* + 1\right)^2 + 1\right\} - \frac{2\sigma}{n^{(1)}} \mu_{(1)} \mu^* + \frac{2\sigma}{N} \bar{\mu}\left(\mu^* + 1\right)\right]$$

$$+ \sigma^2 \frac{N-1}{N-2} \left[\frac{1}{n^{(1)^2}} + \frac{n^{(2)}}{N} e^{-\mu^*} \left(\frac{1}{n^{(2)^2}} - \frac{1}{n^{(1)^2}}\right)\right]$$

$$-2\sigma \left[\left(\mu_{(2)} + \frac{\sigma}{n^{(2)}}\right) \frac{e^{-\mu^*}}{N} + \frac{\sigma}{n^{(1)}} \left(\frac{e^{-\mu^*}}{N} + \frac{1 - e^{-\mu^*}}{n^{(1)}}\right)\right] \qquad \ldots \quad (3.13)$$

4. Comparisons

4.1. Small Sample Comparisons

Since $\left|Bias\left(T_M\right)\right| - \left|Bias(T^*)\right| = \frac{\sigma}{n^{(1)}}\left(1 - e^{-\mu^*}\right) > 0, T^*$ has smaller absolute bias than that of T_M, i.e. T^* is closer to $\mu_{(1)}$ than T_M. This fact, however, can be proved by using Pitman's closeness property which says:

An estimator T is closer to θ than another estimator T' if

$$P_\theta\left[\left|T - \theta\right| < \left|T' - \theta\right|\right] > \frac{1}{2} \qquad \ldots \qquad (4.1)$$

In our case

$$T = Y - \frac{\hat{\sigma}}{\hat{n}^{(1)}}, T' = Y$$

and

$$\theta = \mu_{(1)}.$$

Thus

$$\left|T - \theta\right| < \left|T' - \theta\right| \Leftrightarrow Y - \frac{\hat{\sigma}}{2\hat{n}^{(1)}} > \mu_{(1)}. \quad \ldots \quad (4.2)$$

Now, after some routine steps, it can be shown that

$$P = P\left[Y - \frac{\hat{\sigma}}{2\hat{n}^{(1)}} > \mu_{(1)}\right]$$

$$= Q.P\left(\chi_\nu^2 < c_1\right) + \left(\frac{\nu}{\nu+1}\right)^{\frac{\nu}{2}} \left\{(1-Q) + Qe^{\frac{c_1}{2\nu}} P\left(\chi_\nu^2 > c_2\right)\right\} \ldots (4.3)$$

where

$$\left.\begin{aligned} \nu &= 2N - 4, Q = \frac{n^{(2)}}{N} e^{-\mu^*} \\ c_1 &= \frac{4n^{(2)}(N-2)\left(\mu_{(2)} - \mu_{(2)}\right)}{\sigma} \\ c_2 &= \frac{2n^{(2)}(2N-3)\left(\mu_{(2)} - \mu_{(1)}\right)}{\sigma} \end{aligned}\right\} \qquad \ldots \quad (4.4)$$

Note that $c_1 = \frac{\nu}{\nu+1} c_2$. It is easy to verify that, for N>2

$$\left(\frac{\nu}{\nu+1}\right)^{\frac{\nu}{2}} > \frac{1}{2} \qquad \ldots \quad (4.5)$$

Since

$$c_1 = \frac{\nu}{\nu+1} c_2,$$

let

$$\psi(x) = P)X_\nu^2 < x) + \left(\frac{\nu}{\nu+1}\right)^{\frac{\nu}{2}} e^{x/2\nu} P\left(X_\nu^2 > \frac{\nu+1}{\nu} X\right) \quad \ldots (4.6)$$

which can be written as

$$\psi(x) = 1 - P\left[U > x, V < \frac{U-x}{\nu}\right],$$

where

$$U \sim \chi_\nu^2$$

and

$V \sim \chi_2^2$ independently.

Now, since $x \geq 0$ and $\frac{2}{\nu}\frac{U}{V} \sim F_{\nu,2}$, we get

$$\psi(x) = 1 - P[U - \nu V > x]$$

$$\geq 1 - P[U - \nu V > 0]$$

$$= P\left[F_{\nu,2} < 2\right]$$

$$= \left(\frac{\nu}{\nu+1}\right)^{\frac{\nu}{2}}. \quad \ldots (4.7)$$

(4.5) and (4.7) imply that

$$P > \frac{1}{2} \quad \ldots (4.8)$$

and hence T^* is Pitman closer than T_M

The MSE of T_M is not easy to compare algebraically with that of T^*. We shall, however, consider some asymptotic and numerical comparisons in the later sections.

4.2. Asymptotic Comparisons

It can be easily seen that $Bias(T_M), Bias(T^*), MSE(T_M)$ and $MSE(T^*)$ all tend to zero as $N \to \infty$. Hence T^* and T_M are consistent. Also, we see that

$$\lim_{N\to\infty}\left[N\left|Bias(T_M)\right|\right] = \frac{\sigma}{\lambda^{(1)}} \quad \ldots (4.9)$$

where

$$\lambda^{(1)} = \lim_{N\to\infty} \frac{n^{(1)}}{N}$$

and that

$$\lim_{N\to\infty}\left[N\left|Bias(T^*)\right|\right] = 0. \quad \ldots (4.10)$$

Again, as $N \to \infty$,

$$N^2\left\{MSE(T_M) - MSE(T^*)\right\} \to \frac{\sigma^2}{\lambda^{(1)^2}}. \ldots (4.11)$$

(4.9) and (4.10) imply

$$\lim_{N\to\infty}\left[N\left\{\left|Bias(T_M)\right| - \left|Bias(T^*)\right|\right\}\right] = \frac{\sigma}{\lambda^{(1)}} \quad \ldots (4.12)$$

(4.11) and (4.12) mean that T^* is at least asymptotically a better estimator of $\mu_{(1)}$ than T_M.

5. Case of Unequal Scale

Here we consider the estimators

$$T_M = Y \text{ and } T^* = Y - \frac{\hat{\sigma}^{(1)}}{\hat{n}^{(1)}},$$

where

$$\hat{\sigma}^{(1)} = \hat{\sigma}_i = \frac{S_i}{n_1 - 1} \text{ if } Y = Y_i, i = 1, 2.$$

Then, as in the case of equal scale, we see that

$$Bias(T_M) = \frac{\sigma^{(1)}}{n^{(1)}}\left(1 - e^{-\mu^*}\right) + \left(\sum_{i=1}^2 \frac{n^{(i)}}{\sigma^{(i)}}\right)^{-1} e^{-\mu^*} > 0 \quad \ldots (5.1)$$

$$Bias(T^*) = -\left(\sum_{i=1}^2 \frac{n^{(i)}}{\sigma^{(i)}}\right)^{-1} e^{-\mu^*} < 0 \quad \ldots (5.2)$$

$$MSE(T_M) = \frac{2\sigma^{(1)^2}}{n^{(1)^2}} + e^{-\mu^*}\left[\left(\sum_{i=1}^2 \frac{n^{(i)}}{\sigma^{(i)}}\right)^{-2}\left(\sum_{i=1}^2 \frac{n_i \mu_i}{\sigma_i}\right)^2 - \mu_{(1)}^2\right.$$

$$+\left\{\left(\sum_{i=1}^2 \frac{n^{(i)}}{\sigma^{(i)}}\right)^{-2} - \frac{\sigma^{(1)^2}}{n^{(1)^2}}\right\}\left\{\left(\mu^* + 1\right)^2 + 1\right\}$$

$$-2\mu_{(1)}\left\{\frac{\sigma^{(1)}}{n^{(1)}}\mu^* + \left(\sum_{i=1}^2 \frac{n^{(i)}}{\sigma^{(i)}}\right)^{-1}\right\}$$

$$+2\left\{\left(\sum_{i=1}^{2}\frac{n^{(i)}}{\sigma^{(i)}}\right)^{-2}\left(\sum_{i=1}^{2}\frac{n_i\mu_i}{\sigma_i}\right)\left(\mu^*+1\right)\right\}\Bigg] \quad \ldots(5.3)$$

$$MSE\left(T^*\right)=MSE\left(T_M\right)+2\mu_{(1)}e^{-\mu^*}\left(\sum_{i=1}^{2}\frac{n^{(i)}}{\sigma^{(i)}}\right)^{-1}+\frac{\sigma^{(1)^2}}{n^{(1)}\left(n^{(1)}-1\right)}$$

$$+\left\{\frac{\sigma^{(2)^2}}{n^{(2)}\left(n^{(2)}-1\right)}-\frac{\sigma^{(1)^2}}{n^{(1)}\left(n^{(1)}-1\right)}\right\}e^{-\mu^*}\frac{n^{(2)}}{\sigma^{(2)}}\left(\sum_{i=1}^{2}\frac{n^{(i)}}{\sigma^{(i)}}\right)^{-1}$$

$$-2\Bigg[\left(\mu_{(2)}+\frac{\sigma^{(2)}}{n^{(2)}}\right)\left(\sum_{i=1}^{2}\frac{n^{(i)}}{\sigma^{(i)}}\right)^{-1}e^{-\mu^*}$$

$$+\frac{\sigma^{(1)}}{n^{(1)}}\left\{\frac{\sigma^{(1)}}{n^{(1)}}\left(1-e^{-\mu^*}\right)+\left(\sum_{i=1}^{2}\frac{n^{(i)}}{\sigma^{(i)}}\right)^{-1}e^{-\mu^*}\right\}\Bigg] \quad \ldots(5.4)$$

where

$$\sigma^{(i)}=\sigma_i \text{ if } \mu_{(1)}=\mu_i, i=1,2$$

and $\mu^*=\dfrac{n^{(1)}\left(\mu_{(2)}-\mu_{(1)}\right)}{\sigma^{(1)}}$.

Here

$$\left|Bias(T_M)\right|-\left|Bias(T^*)\right|=\frac{\sigma^{(1)}}{n^{(1)}}\left(1-e^{-\mu^*}\right)>0,$$

and

$$P=P\left[\left|Y-\mu_{(1)}\right|>\left|Y-\frac{\hat{\sigma}^{(1)}}{\hat{n}^{(1)}}-\mu_{(1)}\right|\right]$$

$$=P\left[Y-\frac{\hat{\sigma}^{(1)}}{2\hat{n}^{(1)}}>\mu_{(1)}\right]$$

$$=\left(\frac{v_1}{v_1+1}\right)^{\frac{v_1}{2}}(1-Q)+Q\left\{P\left(\chi^2_{v_2}<c_1\right)+\left(\frac{v_1}{v_1+1}\right)^{\frac{v_1}{2}}e^{\frac{c_1}{2v_2}}P\left(\chi^2_{v_2}>c_2\right)\right\}$$

$$\ldots(5.5)$$

where

$$\left.\begin{aligned} v_1&=2\left(n^{(1)}-1\right), v_2=2\left(n^{(2)}-1\right)\\ Q&=\frac{n^{(2)}}{\sigma^{(2)}}\left(\sum_{i=1}^{2}\frac{n^{(i)}}{\sigma^{(i)}}\right)^{-1}e^{-\mu^*}\\ c_1&=\frac{4n^{(2)}\left(n^{(2)}-1\right)\left(\mu_{(2)}-\mu_{(1)}\right)}{\sigma^{(2)}}\\ c_2&=\frac{2n^{(2)}(2n^{(2)}-1)\left(\mu_{(2)}-\mu_{(1)}\right)}{\sigma^{(2)}}=\frac{v_2+1}{v_2}c_1\end{aligned}\right\} \quad \ldots(5.6)$$

As in section 4.1, it can be similarly shown that $P>\dfrac{1}{2}$ and hence T^* is Pitman closer than T_M Again,

$$\lim_{N\to\infty}\left\{N\left(\left|Bias\left(T_M\right)\right|-\left|Bias(T^*)\right|\right)\right\}=\frac{\sigma^{(1)}}{\lambda^{(1)}}$$

and

$$\lim_{N\to\infty}\left[N^2\left\{MSE\left(T_M\right)-MSE(T^*)\right\}\right]=\frac{\sigma^{(1)^2}}{\lambda^{(1)^2}}$$

implying that T^* is also asymptotically better than T_M

6. Numerical Results

Here we write

$$B_1=Bias\left(T_M\right), \qquad B_2=Bias\left(T^*\right)$$

$$M_1=MSE\left(T\quad\right), \qquad M_2=MSE\left(T^*\right).$$

From the table 6.1, we observe that :

(i) Both $Bias(\quad)$ and $Bias\left(T^*\right)$ decrease with the increase in sample sizes, and increase with the increase in σ, as is evident from their theoretical expressions.

(ii) The mean square errors, too, decrease with the increase in sample sizes, and increase with the increase in σ.

(iii) As has been already algebraically established *Bias*(T^*) is always less than *Bias*(T_M) absolutely. But what is more important is that *MSE*(T^*) is less than *MSE*(T_M)

7. Concluding Remarks

Comparison of the proposed estimator with another existing estimator $T_U=\min_i=1,2\left\{Y_i-\dfrac{\hat{\sigma}}{n_i}\right\}$ is left undone because as in the case of proposed estimator or the ML estimator, we cannot have exact mathematical expressions for the performance criteria of T_U. However, we can get a fair idea of the performance of T^* over T_U through simulation studies. The case of equal sample sizes follows easily from the above study as a particular case.

Table 6.1: Calculations of B_1,B_2,M_1 and M_2 based on 10,000 simulations

(n_1, n_2)	σ	(μ_1, μ_2)	μ^*	$\lvert B_1 \rvert$	$\lvert B_2 \rvert$	M_1	M_2
(10,12)	1	(1,1.5)	5	0.0996	0.0003	0.0195	0.0102
		(1.5,2.5)	10	0.01	2.06×10^{-6}	0.01999	0.01049
		(2,1)	12	0.083	2.79×10^{-7}	0.013885	0.007287
	1.5	(1,1.5)	3.33	0.147	2.44×10^{-2}	0.0408	0.01713
		(1.5,2.5)	6.67	0.15	8.65×10^{-5}	0.0447	0.0232
		(2,1)	8	0.125	2.29×10^{-5}	0.0296	0.0148
	2	(1,1.5)	2.5	0.191	0.00746	0.0658	0.0209
		(1.5,2.5)	5	0.199	0.0006	0.0781	0.0389
		(2,1)	6	0.166	0.0002	0.0432	0.0173
(20,25)	1	(1,1.5)	10	0.05	1.01×10^{-6}	0.004999	0.00256
		(1.5,2.5)	20	0.05	4.58×10^{-11}	0.005	0.00256
		(2,1)	25	0.04	3.09×10^{-12}	0.0032	0.0016
	1.5	(1,1.5)	6.67	0.0749	4.2×10^{-5}	0.0112	0.00565
		(1.5,2.5)	13.33	0.075	5×10^{-2}	0.01125	0.00576
		(2,1)	16.67	0.06	1.9×10^{-9}	0.0072	0.0037
	2	(1,1.5)	5	0.0996	2.99×10^{-4}	0.0195	0.00946
		(1.5,2.5)	10	0.01	2.01×10^{-6}	0.01999	0.0102
		(2,1)	12.5	0.08	1.66×10^{-7}	0.01278	0.0065
	5	(1,1.5)	2	0.2312	0.015	0.0926	0.0182
		(1.5,2.5)	4	0.002	0.002	0.118	0.0532
		(2,1)	5	0.00075	0.00075	0.0465	0.0088

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