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2^n - Factorial Set Up – Some Aspects

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ABSTRACT

Symbolic direct product of vectors and direct product of matrices have been used to represent treatment combinations and effect contrasts in 2^n - factorial set-up. Relations with Hadamard matrices with the coefficient matrices of the factorial effects are shown and also the inherent algebraic results are established for Yates' algorithm. Also the two symbol orthogonal arrays obtainable from control blocks of confounded 2^n designs are demonstrated.

1. Introduction

This is an expository article with a view to provide some insight into two level factorial set-up. The results are available in one or other form in literature. But an attempt has been made to discuss them under one general set-up giving the mathematical basis to some usual practices of computing the factorial effects and the control blocks.

Consider the following 2^2 - factorial set-up with factors A_1 and A_2 where each factor has two levels indexed by 0 and 1.

2^2 – Factorial set-up

Factors	A_1	A_2
Levels	(0, 1)	(0, 1)

The 2^2 level combinations can be obtained as symbolic direct product of the vector (0, 1) with (0, 1) *i.e.*

$(0, 1) * (0, 1) = (00, 10, 01, 11) = (x_1, x_2);$

$x_i = 0, 1$ for all i (1.1)

where '*' denotes the symbolic direct product (SDP)

The treatment effect at the level combination (x_1, x_2) is denoted by t (x_1, x_2) or by $a_1^{x_1} a_2^{x_2}$. The vector of treatment effects in standard order, *i.e.*, the order in (1.1)) is given by

$t = \{ t(00), t(10), t(01), t(11)\}$

$= \{a_1^0 a_2^0, a_1^1 a_2^0, a_1^0 a_2^1, a_1^1 a_2^1\}$

$= (1, a_1, a_2, a_1 a_2)$ (1.2)

Here, (1.2) is written by assuming the treatment symbol $a_1^{x_1} a_2^{x_2}$ as product of two real numbers $a_1^{x_1}$ and $a_2^{x_2}$ where x's are playing the role of power. In general, for 2^n set-up, the level combinations and treatment effects on them can be written as follows :

Level combination :

$(0, 1) * (0, 1) *…* (0, 1) = (00, 10, 01, 11) * (01) *…* (0,1)$

$= (000, 100, 010, 110, 001, 101, 011, 111) *…. * (0, 1).$

$= (00...0, 100...0, ……, 111…1)$

$= \{(x_1, x_2,…, x_n);$ $x_i = 0, 1; i =1, 2….n\}$ (1.3)

Treatment effects

$= \{t (000…), t (10…0)….t (1, 1…1)\}$

$= \{a_1^0 a_2^0 ….a_n^0, a_1^1 a_2^0.. a_n^0,….., a_1^{x_1} a_2^{x_2} …a_n^{x_n}\}$

$= (1, a_1, a_2, a_1a_2, a_3, a_1a_3, ……, a_1a_2.......a_n)$ (1.4)

(1.4) is written by dropping the letters with even power. So the use of SDP gives the treatment effects in the same order as is obtained in the usual way of writing the treatment effects by product of letters one after another.

2. Factorial Effects

The factorial effects are defined in the following way. Consider the 2^2 – factorial set- up with factors as A_1 and A_2. The effect of A_1 for changing level 0 to 1 when A_2 is fixed at the level 0 (called simple effect of A_1) is given by $a_1^1 a_2^0 - a_1^0 a_2^1$. Similarly the simple effect of A_1 when A_2 is at level 1is given by $a_1^1 a_2^1 - a_1^0 a_2^1$. This can be displayed as

Levels of A_2	Simple effects of A_1
0	$-a_1^0 a_2^0 + a_1^1 a_2^0$
1	$-a_1^0 a_2^1 + a_1^1 a_2^1$

A_1 = Main effect (m.e) of A_1

= total of the simple effects of A_1 over the levels of A_2

$= (-a_1^0 a_1^1 + a_1^1 a_2^0) + (-a_1^0 a_1^1 + a_1^1 a_2^1)$

$= (-a_1^0 + a_1^1) (a_2^0 + a_2^1)$

$= (-1 + a_1) (1 + a_2)$ (2.1)

(2.1) is obtained from the usual practice of assuming the treatment effects as product of powers of a's. Similarly in the same way

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A_2 = Main effect (m.e) of A_2

= sum of the simple effects of A_2 over the levels A_1

= $(1 + a_1)(-1 + a_2)$ (2.2)

A_1A_2 = Interaction effects of A_1 and A_2

= Difference of the simple effects of A_1 at difference level of A_2

= $(-1 + a_1)(-1 + a_2)$ (2.3)

I = Total of 2^2 treatment effects

= $a_1^0a_2^0 + a_1^1a_2^0 + a_1^0a_2^1 + a_1^1a_2^1$

= $(1 + a_1)(1 + a_2)$ (2.4)

The results (2.1) - (2.4) give an easy way of writing the factorial effects though the factorizations have no meaning. They should be expanded and treatment effects are to be put in the corresponding level combinations.

Following the same artificial representation the factorial effects in 2^2 – set-up can be written as $A_1^0A_2^0$ for I, $A_1^1A_2^0$ for the m.e of A_1, $A^0A_2^1$ for the m.e of A_2 and $A_1^1A_2^1$ for the interaction effect of A_1A_2. So, in general, any effect can be denoted by $A_1^{\alpha_1}A_2^{\alpha_2}$, where α_i = 0, 1and i=1, 2. In tabular form, the effects can be represented as:

Table 2.1

(α_1, α_2)	Effects ($A_1^{\alpha_1}A_2^{\alpha_2}$)
0 0	$A_1^0A_2^0 = I$
1 0	$A_1^1A_2^0 = A_1$
0 1	$A_1^0A_2^1 = A_2$
1 1	$A_1^1A_2^1 = A_1A_2$

From Table 2.1 and (2.1) - (2.4) we see that the factorial effects can be expressed generally in terms of the treatments effects as

$$A_1^{\alpha_1}A_2^{\alpha_2} = \{(-1)^{\alpha_1}a_1^0 + a_1^1\}\{(-1)^{\alpha_2}a_2^0 + a_2^1\}$$

$$= \{\sum_{x_1=0}^{1}(-1)^{\alpha_1(1-x_1)}a_1^{x_1}\}\{\sum_{x_2=0}^{1}(-1)^{\alpha_2(1-x_2)}a_2^{x_2}\}$$

$$= \sum_{x_1=0}^{1}\sum_{x_2=0}^{1}(-1)^{\alpha_1(1-x_1)+\alpha_2(1-x_2)}.a_1^{x_1}a_2^{x_2}. \qquad (2.5)$$

In general for a 2^n- factorial set-up, it follows that any factorial effect can be denoted by $A_1^{\alpha_1}A_2^{\alpha_2}...A_n^{\alpha_n}$ (including the total) α_i = 0,1; i =1,2,…,n and it can be shown that $A_1^{\alpha_1}A_2^{\alpha_2}...A_n^{\alpha_n}$ can be represented in terms of the treatment effects $a_1^{x_1}a_2^{x_2}...a_n^{x_n}$, x_i = 0,1; i = 1, 2, ….., n as

$$A_1^{\alpha_1}A_2^{\alpha_2}...A_n^{\alpha_n} = \sum_{x_1=0}^{1}\sum_{x_2=0}^{1}...\sum_{x_n=0}^{1}(-1)^{\sum\alpha_i(1-x_i)}.a_1^{x_1}a_2^{x_2}...a_n^{x_n}$$

$$= \prod_{i=1}^{n}\{\sum_{x_i=0}^{1}(-1)^{\alpha_i(1-x_i)}a_i^{x_i}\} \qquad (2.6)$$

3. Algorithm for expressing the factorial effects in terms of treatment effects.

A factorial effect is said to be even (odd) if the number of capital letters in the effect is even (odd). In 2^2-factorial set-up $A_1 = A_1^1A_2^0$ is an odd effect while the effect $A_1A_2 = A_1^1A_2^1$ is even. Consider the following table.

Table 3.1

Effect	Nature of the FE	$\sum\alpha_i$
$A_1 = A_1^1A_2^0$	Odd	1+0=1
$A_2 = A_1^0A_2^1$	Odd	0+1=1
$A_1A_2 = A_1^1A_2^1$	Even	1+1+0 (mod2).

Generalizing the above fact it is easily seen that for 2^n-factorial set-up, a FE $A_1^{\alpha_1}A_2^{\alpha_2}...A_n^{\alpha_n}$ is odd if $\sum_{i=1}^{n}\alpha_i = 1$ (mod 2) and is even if $\sum_{i=1}^{n}\alpha_i = 0$ (mod 2). Again the number of letters common between the FE $A_1^{\alpha_1}A_2^{\alpha_2}...A_n^{\alpha_n}$ and the treatment effect $a_1^{x_1}a_2^{x_2}...a_n^{x_n}$ is given by $\sum_{i=1}^{n}\alpha_i x_i$ as α_i's and x_i's are 0 or 1. So the number of letters common between $A_1^{\alpha_1}A_2^{\alpha_2}...A_n^{\alpha_n}$ and $a_1^{x_1}a_2^{x_2}...a_n^{x_n}$ is even if $\sum_{1}^{n}\alpha_i x_i = 0$ (mod 2) and is odd if $\sum_{1}^{n}\alpha_i x_i = 1$ (mod 2). From (2.6) and the above discussions it follows that the coefficient of the treatment effect $a_1^{x_1}a_2^{x_2}...a_n^{x_n}$ in the FE $A_1^{\alpha_1}A_2^{\alpha_2}...A_n^{\alpha_n}$ is

$(-1)^{\sum \alpha_i(1-x_i)} = (-1)^{\sum \alpha_i}(-1)^{-\sum \alpha_i x_i}$. This is 1 if $\sum_1^n \alpha_i = 0\ (1)$ (mod 2) and $\sum_1^n \alpha_i x_i = 0\ (1)$ (mod 2) *i.e.* if the FE is even (odd) and the number of letters common between the FE and the treatment effect is even (odd). We consider all the cases and construct the following factorial table.

Table 3.2

Nature of the FE $A_1^{\alpha_1} A_2^{\alpha_2} \ldots A_n^{\alpha_n}$	Number of letters common between the FE and the treatment effect $a_1^{x_1} a_2^{x_2} \ldots a_n^{x_n}$	Coefficient of $a_1^{x_1} a_2^{x_2} \ldots a_n^{x_n}$ in the expansion of $A_1^{\alpha_1} A_2^{\alpha_2} \ldots A_n^{\alpha_n}$
Even $\left(\sum \alpha_i = 0,\ \text{mod } 2\right)$	Even $\left(\sum \alpha_i x_i = 0, \text{mod } 2\right)$	$(-1)^0(-1)^0 = 1$
Even $\left(\sum \alpha_i = 0,\ \text{mod } 2\right)$	Odd $\left(\sum \alpha_i x_i = 1, \text{mod } 2\right)$	$(-1)^0(-1)^1 = -1$
Odd $\left(\sum \alpha_i = 1,\ \text{mod } 2\right)$	Even $\left(\sum \alpha_i x_i = 0, \text{mod } 2\right)$	$(-1)^1(-1)^0 = -1$
Odd $\left(\sum \alpha_i = 1,\ \text{mod } 2\right)$	Odd $\left(\sum \alpha_i x_i = 1, \text{mod } 2\right)$	$(-1)^1(-1)^1 = 1$

The above discussions give the underlying mechanism of the usually practiced thumb rule of signs in 2^n- factorial set-up.

Also from (2.6) it can be verified that any two distinct FE's are mutually orthogonal. Consider two FE's $A_1^{\alpha_1} A_2^{\alpha_2} \ldots A_n^{\alpha_n}$ and $A_1^{\beta_1} A_2^{\beta_2} \ldots A_n^{\beta_n}$. As they are distinct, at least for one i, $\alpha_i \neq \beta_i$. Without loss of generality we assume that $\alpha_i = 0$, $\beta_i = 1$. Then the inner product of the coefficients is given by

$$\prod_{i=1}^{n} \left\{ \sum_{x_i=0}^{1} (-1)^{\alpha_i(1-x_i)} . (-1)^{\beta_i(1-x_i)} \right\}$$

$$= \prod_{i=1}^{n} \left\{ (-1)^0 (-1+1) \right\} = 0$$

Also for $(\alpha_1, \alpha_2, \ldots \alpha_n) \neq (0, 0, ..0)$ it can be easily seen that the sum of the coefficients in the FE $A_1^{\alpha_1} A_2^{\alpha_2} \ldots A_n^{\alpha_n}$ is zero. So the factorial effect $A_1^{\alpha_1} A_2^{\alpha_2} \ldots A_n^{\alpha_n}$ is a treatment contrast.

4. Relation between the coefficient matrix and Hadamard matrix of order 2^n

We consider the following table giving the coefficient matrix in the 2^2- factorial set-up.

Table 4.1

FE	$a_1^0 a_2^0$	$a_1^1 a_2^0$	$a_1^0 a_2^1$	$a_1^1 a_2^1$	
	Treatment effects				
$A_1^0 A_2^0 = I$	1	1	1	1	
$A_1^1 A_2^0 = A_1$	-1	1	-1	1	= A
$A_1^0 A_2^1 = A_2$	-1	-1	1	1	
$A_1^1 A_2^1 = A_1 A_2$	1	-1	-1	1	

Here we see that the elements of the coefficient matrix A is +1 or -1 and any two rows and any two columns are orthogonal. Such a square matrix is called a Hadamard matrix. Generalizing the above, it can be seen for the 2^n- case that the coefficient matrix is a Hadamard matrix of order 2^n.

FE **Treatment effects**

$$A_1^{\alpha_1} A_2^{\alpha_2} \cdot \updownarrow \ldots A_n^{\alpha_n} \quad \begin{bmatrix} \leftarrow a_1^{x_1} a_2^{x_2} \cdot \downarrow \ldots a_n^{x_n} \rightarrow \\ \rightarrow (-1)^{\Sigma \alpha_i (1-x_i)} \\ \\ \end{bmatrix} = H_{2^n}$$

The coefficient $(-1)^{\sum \alpha_i (1-x_i)}$ of $a_1^{x_1} a_2^{x_2} \ldots . a_n^{x_n}$ in $A_1^{\alpha_1} A_2^{\alpha_2} \ldots . A_n^{\alpha_n}$ is the element of H_{2^n}, a Hadamard matrix of order 2^n in the $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ th row and $(x_1, x_2, \ldots, x_n)$ th column.

Note that the above Hadamard matrix is actually given by $H_2 \otimes H_2 \otimes ..... \otimes H_2$ where $\otimes$ denotes Kronecker Product and $H_2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

5. Yates' Algorithm

There is another schematic method known as Yates' algorithm of obtaining the factorial effects in 2^n- factorial set-up in terms of the treatment effects. The algorithm is demonstrated below :

Table 5.1 (2^2- set-up)

Treatment effects	Step 1	Step 2	FE
1	$1+a_1$	$1+a_1+a_2+a_1a_2$	I
a_1	$a_2+a_1a_2$	$-1+a_1-a_2+a_1a_2$	A_1
a_2	a_1-1	$-1-a_1+a_2+a_1a_2$	A_2
a_1a_2	$a_1a_2-a_2$	$1-a_1-a_2+a_1a_2$	$A1A_2$

Note that the elements in Step1 and Step2 are given respectively by

$$\begin{pmatrix} 1+a_1 \\ a_2+a_1a_2 \\ -1+a_1 \\ -a_2+a_1a_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ a_1 \\ a_2 \\ a_1a_2 \end{pmatrix} \quad (5.1)$$

and

$$\begin{pmatrix} 1+a_1+a_2+a_1a_2 \\ -1+a_1-a_2+a_1a_2 \\ -1-a_1+a_2+a_1a_2 \\ 1-a_1-a_2+a_1a_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 \\ a_1 \\ a_2 \\ a_1a_2 \end{pmatrix} \quad (5.2)$$

Note that

$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = H_{2^2}, \quad (5.3)$$

where

D_1 = Diag (1,1;1,1)

D_2 = Diag (-1,1;-1,1) (5.4)

Generalizing the above, we get for 2^n- set-up the elements at Step R as

$$\begin{pmatrix} D \\ D^* \end{pmatrix}^R t, \; R = 1, 2, \ldots, n \quad (5.5)$$

Where

D = Diag (1, 1; 1, 1;…; 1,1)

D* = Diag (-1, 1;-1, 1;….,-1, 1)

D's are $2^{n-1} \times 2^n$ matrices each and $t = (1, a_1, a_1a_2, \ldots, a_1a_2 \ldots a_n)'$ is the vector of 2^n treatment effects.

Note that

$$\begin{pmatrix} D \\ D^* \end{pmatrix}^n = H_2 \otimes H_2 \otimes ...... \otimes H_2 = H_{2^n}.$$

The above form of expression of H_{2^n} is the basis of Yates' algorithm.

6. Confounding

The general theory of confounding the factorial effect $A_1^{\alpha_1} A_2^{\alpha_2} .... A_n^{\alpha_n}$ is to allocate 2^n treatments into two blocks B_0 and B_1 each containing 2^{n-1} treatments, where the treatments in B_0 and B_1 are given respectively by

$B_0 = \{(x_1, x_2, \ldots x_n)/\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0 \text{ (mod2)}\}$

$B_1 = \{(x_1, x_2, \ldots x_n)/\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 1 \text{ (mod2)}\}$ (6.1)

See that all the 2^{n-1} treatments in the control block B_0 are even with the confounded effect $A_1^{\alpha_1} A_2^{\alpha_2} .... A_n^{\alpha_n}$. Also all the treatments in B_1 are odd with the confounded effect. Let a level combination $(x_1, x_2, \ldots x_n) \in B_0$ and another level combination $(y_1, y_2 \ldots y_n) \in B_1$ then the level combination $(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$ mod 2 also belongs to B_1 because

$$\sum_{i=1}^{n} \alpha_i (x_i + y_i) = \sum_i \alpha_i x_i + \sum_i \alpha_i y_i = 1 \text{ (mod2) as} \sum_i \alpha_i x_i = 0.$$

So all the treatments in B_1 can be obtained by adding any treatment $(y_1, y_2 \ldots y_n)$ in B_1 to all treatments in B_0.

Example 6.1

Suppose $A_1A_2A_3$ is to be confounded when a 2^3 experiment is conducted in 2 blocks. See that (000, 110, 011, 101) = (1, a_1a_2, a_2a_3, a_1a_3) are even with $A_1A_2A_3$. So they constitute the control block B_0. See that (000, 110, 011, 101) ⊕ (100) = (100, 010, 111, 001) = (a_1, a_2, a_3, $a_1a_2a_3$) = B_0*a_1, which constitute the block B_1.

In the above design $A_1A_2A_3$ is confounded with blocks. This means that the orthogonal contrast representing $A_1A_2A_3$ has expectation $A_1A_2A_3$ + block contrast.

Let y_i, (x_1, x_2, x_3) = observation in the i^{th} block corresponding to the level combination

(x_1, x_2, x_3), i=0, 1

$$= m + b_i + a_1^{x_1} a_2^{x_2} a_3^{x_3} + e_{i,(x_1x_2x_3)} \qquad (6.2)$$

where m = general effect, b_i = i^{th} block effect, i= 0, 1

$a_1^{x1}a_2^{x2}a_3^{x3}$ = treatment effect due to the level combination ($x_1x_2x_3$)

$e_{i, (x1x2x3)}$ = random error with usual assumption

$E[(y_{0(100)} + y_{0(010)} + y_{0(001)} + y_{0(111)}) - (y_{1(000)} + y_{1(110)} +$
$y_{1(101)} + y_{1(011)})]$

$= 4(b_1 - b_0) + A_1A_2A_3 \qquad (6.3)$

FE $A_1A_2A_3$ is not separately estimable. Linear combination of block contrast and $A_1A_2A_3$ is estimable.

On the other hand the expectation of observational contrast for ME A_1 is given by

$$E\{(-y_{0(000)} + y_{0(110)} - y_{0(011)} + y_{0(101)}) + (y_{1(000)} - y_{1(010)} - y_{1(001)} + y_{1(111)})\} = A_1 \qquad (6.4)$$

ME A_1 is estimable free of block contrast. So it is not confounded.

7. Relations with group theory

Consider a 2^3-set-up and let the 2^3 combinations be denoted by their treatment symbols. Therefore G, the set of 8 level combinations is given by

$$G = (1, a_1, a_2, a_1a_2, a_3, a_1a_3, a_2a_3, a_1a_2a_3) \qquad (7.1)$$

Define the binary product * among the elements x, y, ∈ G where the operation is ordinary multiplication with squared letters dropped. Note that

(i) x * y ∈ G ∀ x, y ∈ G

(ii) x * y = y * x

(iii) x * 1 = 1 * x = x ∈ G ; ∀ x ∈ G

(iv) ∃ a y ∈ G for every x in G such that x * y = y * x = 1

Therefore G is a commutative group with identity element 1. Also every element is its inverse.

Also note that the control block B_0 of section 6 is also a subgroup in G. The other block B_1 is obtained by a applying the same operation * on the elements of B_0 by any element of B_1. So B_1 is a coset. As the group is commutative, the right and left cosets are the same.

In general, in a (2^n,2^k) confounded design the set G of all 2^n treatments from a commutative group and the control block B_0 containing 2^{n-k} treatments is a subgroup G. The other (2^k-1) blocks are cosets of B_0 in G.

8. Relation with Orthogonal Arrays (OA) with two symbols in each row

An n × N matrix A with two elements 0 and 1 in every row is said to be an orthogonal array (OA) of strength t (≥ 2) if for the choice of any t rows, all possible 2^t vectors containing 0 and 1 occur an equal number (λ ≥ 1) of times in the t × N sub-matrix as columns. This is denoted by OA [N, n, 2, t]

Example 8.1

Consider the (2^3, 2) design confounding $A_1A_2A_3$. B_0, the control block is given by (1, a_1a_2, a_2a_3, a_1a_3). We represent the 4 level combinations as 4 columns of the array A given by

$$A = \begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$ which is an OA [4, 3, 2, 2]

Replace 0's by -1's in A and adjoin a row (1, 1, 1, 1) and get

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix} = H = A_{4\times4}$$ Hadamard matrix

Maximum number of rows in on OA [N, K, 2, 2] is N-1 *i.e.* max k=N-1. Such an OA is called a saturated OA. From such saturated two symbol array, an N × N Hadamard matrix can always be constructed by the above method.

Example 8.2

Consider the control block of the (2^4, 2) design confounding $A_1A_2A_3A_4$. The treatments combinations are arranged in a 4 × 8 array as before.

$$B_0 = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

See that B_0 is an OA [8, 4, 2, 3]

Example 8.3

The array obtained from the control block of $(2^5, 2^2)$ design confounding $A_1A_2A_3$, $A_3A_4A_5$ is given by

$$B_0 = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{matrix} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} = \text{OA } [8, 5, 2, 2]$$

From the above examples we see that the control block of a confounded $(2^n, 2^k)$ design always gives an OA [N, n, 2, t] where $N = 2^{n-k}$ and the strength t= minimum number of letters in the confounded effects minus one.

9. Concluding Remarks

It is seen that the 2^n factorial structure and the factional designs there on have very interesting algebraic and combinatorial properties.

For further studies the readers may go through the ‘Suggested Readings’ listed below.

10. Suggested Readings

1. Fisher, R.A. (1942): The theory of confounding in factorial experiments in relation to the theory of groups. *Ann. Eugenics* **11 :** 341-53.
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