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On estimation of the PMF and CDF of the logarithmic series distribution

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ABSTRACT

Different methods of estimation of the probability mass function and the cumulative distribution function for the logarithmic series distribution are considered. Following estimation methods are attempted: maximum likelihood estimator, uniformly minimum variance unbiased estimator, plug-in uniformly minimum variance unbiased estimator, least squares estimator, weighted least squares estimator. Monte Carlo simulations are performed to compare the performances of the proposed estimators. A real data set has been analyzed for illustrative purpose.

***Keywords* :** Maximum likelihood estimator; Uniformly minimum variance unbiased estimator; Plug-in uniformly minimum variance unbiased estimator; Least squares estimator; Weighted least squares estimator.

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1. INTRODUCTION

A random variable X is said to have the logarithmic series distribution, if its probability mass function (PMF) is given by

$$P(X=x)=f(x)=\frac{-1}{\ln(1-p)}\frac{p^x}{x},\ x=1,2,\ldots;0<p<1 \tag{1}$$

and its cumulative distribution function (CDF) is given by

$$F(x)=\sum_{w=1}^{x}\frac{-1}{\ln(1-p)}\frac{p^w}{w},\ x=1,2,\ldots;0<p<1. \tag{2}$$

The above distribution has many applications in biology and ecology. It is also used for modelling data linked to the number of species.

Now-a-days researchers have given attention for study of properties and inference on this distribution. Statisticians are most of the times interested in inferring the parameter(s) involved in the distribution. Maximum likelihood estimator (MLE), uniformly minimum variance unbiased estimator (UMVUE) and Bayes estimator of the parameter have been focused by the authors.

Simulated mean squared error of MLE and UMVUE of the parameter p for different sample size n has been shown in fig. 1. Theoretical mean squared errors (MSEs) are not studied as no closed form expression of MLE is available in this case. However, from fig. 1, it is noticed that UMVUE is better than MLE of p in MSE sense. Hence it seems to be an interesting study whether the same or different preservation rule prevails for the MLE, UMVUE and plug-in-UMVUE (PUMVUE) of the PMF and the CDF of this distribution.

We see many situations where we have to estimate PMF, CDF or both. For instance, PMF can be used for estimation of differential entropy, Renyi entropy, Kullback-Leibler divergence and Fisher information; CDF can be used for estimation of the Survival/Reliability function, mean residual (inactivity) life function, cumulative residual entropy, the quantile function, Bonferroni curve, Lorenz curve, and both the PMF and the CDF can be used for estimation of probability weighted moments, hazard rate function, mean deviation about mean etc.

Some studies on the estimation of the probability density function (PDF) or PMF and CDF have appeared in recent literature for some distributions: [1], [2], [3, 4, 5], [6, 7], [8], [9, 10, 11], [12] among others.

Following is the organization of the paper. Section 2 discusses the MLEs of the PMF and the CDF. Section 3 devotes to derive UMVUEs of the same. Section 4 discusses estimators based on plug-in UMVUE (PUMVUE). Section 5 takes into account least squares and weighted least squares estimators. Section 6 shows reliability and its related measures. All the estimators have been compared through simulation study in section 7. A data set has been analyzed and summary result has been reported in section 8. Section 9 concludes.

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2. MLE of the PMF and the CDF

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the logarithmic series distribution given by (1). The MLE of the above distribution is being derived as follows. The likelihood function of p is given by

$$\begin{aligned} L(p;x) &= \left(\frac{-1}{\ln(1-p)}\right)^n p^{\sum_{i=1}^n X_i} \frac{1}{\prod_{i=1}^n X_i} \\ \text{so, } l(p) &= \ln L(p;x) \\ &= -n\ln(-\ln(1-p)) + \sum_{i=1}^n X_i \ln p - \sum_{i=1}^n \ln X_i \end{aligned}$$

Now,

$$\begin{aligned} \frac{dl(\theta)}{d\theta} &= 0 \\ \Rightarrow \frac{n}{(1-p)\ln(1-p)} &= -\frac{1}{p}\sum_{i=1}^n X_i \\ \Rightarrow \frac{1}{\ln(1-p)^{\frac{1-p}{p}}} &= -\frac{T}{n} \\ \Rightarrow (1-p)^{\frac{1-p}{p}} &= e^{-\frac{n}{T}}, \end{aligned} \tag{3}$$

where, $T = \sum_{i=1}^n X_i$.

Here we cannot find out any closed form expression for the MLE of p. Therefore, by using numerical approach, we have to find out the root of the equation. By invariance property, we have the MLE of the PMF and the CDF by substituting the MLE of p.

3. UMVUE of the PMF and the CDF

In this section, we obtain the UMVUE of the PMF and the CDF of the logarithmic series distribution. Also, we obtain the MSEs (Variances) of these estimators.

Here $T = \sum_{i=1}^n X_i$ is a complete sufficient statistic for p. Being a sum of n independent random variables with logarithmic series distribution having the parameter p has Stirling distribution of the first kind *SDFK*(n, p) [13], T has the following mass function

$$g(t;n,p) = P(T=x) = \frac{n!|s(x,n)|p^x}{x!(-\ln(1-p))^n},\ x = n, n+1, \ldots \tag{4}$$

Here $s(x, n)$ is the Stirling function of the first kind. According to Rao-Blackwell, and Lehmann-Scheffe theorems, we get the UMVUE of the PMF and the CDF as follows :

Consider,

$$\begin{aligned} Y &= 1 \text{ if } X_1 = k; k = 1, 2, \ldots \\ &= 0 \text{ otherwise} \end{aligned}$$

Then

$$E_p(Y) = P_p[X_1 = k] = \gamma(p) \text{ for all } p.$$

where $\gamma(p) = \frac{-1}{\ln(1-p)}\frac{p^k}{k}$. Therefore,

$$\begin{aligned} E[Y|t] &= P_p[X_1 = k | T = t] \\ &= \frac{P_p[X_1 = k,\ \sum_{i=2}^n X_i = t-k]}{P_p[T=t]} \\ &= \frac{1}{nk} \times \frac{|s(t-k, n-1)|}{|s(t,n)|} \times \frac{t!}{(t-k)!};\ k = 1, \ldots, t \end{aligned} \tag{5}$$

Hence, E[$Y|t$] gives the UMVUE of the PMF of the logarithmic series distribution.

Theorem 3.1.

Let T = t be given. Then

$$\widehat{f}(x) = \frac{1}{nx} \times \frac{|s(t-x, n-1)|}{|s(t,n)|} \times \frac{t!}{(t-x)!}; \; x = 1, \ldots, (t-n+1) \tag{6}$$

is UMVUE for f(x) and

$$\widehat{F}(x) = \sum_{w=1}^{x} \frac{1}{nw} \times \frac{|s(t-w, n-1)|}{|s(t,n)|} \times \frac{t!}{(t-w)!}; \; x = 1, \ldots, (t-n+1) \tag{7}$$

is UMVUE for F(x).

Proof. The proof of $\hat{f}(x)$ is the UMVUE follows from (5). The proof that $\hat{F}(x)$ is the UMVUE follows by summing up $\hat{f}(x)$.

The variance of $\hat{f}(x)$ is given by

$$\begin{aligned}
Var(\widehat{f}(x)) &= E(\widehat{f}(x))^2 - E^2(\widehat{f}(x)) \\
&= \sum_{t=max(n,x+n-1)}^{\infty} (\widehat{f}(x))^2 g(t;n,p) - \left[\sum_{t=max(n,x+n-1)}^{\infty} \widehat{f}(x) g(t;n,p) \right]^2 \\
&= \left[\sum_{t=max(n,x+n-1)}^{\infty} \frac{1}{n^2x^2} \times \frac{|s(t-x,n-1)|^2}{|s(t,n)|^2} \times \frac{t!^2}{(t-x)!^2} \times \frac{n!|s(t,n)|p^t}{t!(-\ln(1-p))^n} \right] \\
&\quad - \left[\sum_{t=max(n,x+n-1)}^{\infty} \frac{1}{nx} \times \frac{|s(t-x,n-1)|}{|s(t,n)|} \times \frac{t!}{(t-x)!} \times \frac{n!|s(t,n)|p^t}{t!(-\ln(1-p))^n} \right]^2 \\
&= \frac{(n-1)!}{x^2[-\ln(1-p)]^n} \left[\frac{1}{n} \sum_{t=max(n,x+n-1)}^{\infty} \frac{|s(t-x,n-1)|^2}{|s(t,n)|} \times \frac{t!p^t}{((t-x)!)^2} \right. \\
&\quad \left. - \frac{(n-1)!}{(-\ln(1-p))^n} \left\{ \sum_{t=max(n,x+n-1)}^{\infty} \frac{p^t|s(t-x,n-1)|}{(t-x)!} \right\}^2 \right],
\end{aligned}$$

and the variance of $\hat{F}(x)$ is given by

$$\begin{aligned}
Var(\widehat{F}(x)) &= \frac{(n-1)!}{(-\ln(1-p))^n} \left[\frac{1}{n} \sum_{t=max(n,x+n-1)}^{\infty} \frac{t!p^t}{|s(t,n)|} \left\{ \sum_{w=1}^{x} \frac{|s(t-w,n-1)|}{w(t-w)!} \right\}^2 - \frac{(n-1)!}{(-\ln(1-p))^n} \right. \\
&\quad \left. \times \left\{ \sum_{t=max(n,x+n-1)}^{\infty} \sum_{w=1}^{x} \frac{p^t|s(t-w,n-1)|}{w(t-w)!} \right\}^2 \right].
\end{aligned}$$

The Graph of variance of UMVU estimator of the PMF and the CDF for different sample sizes is shown in Figure 2.

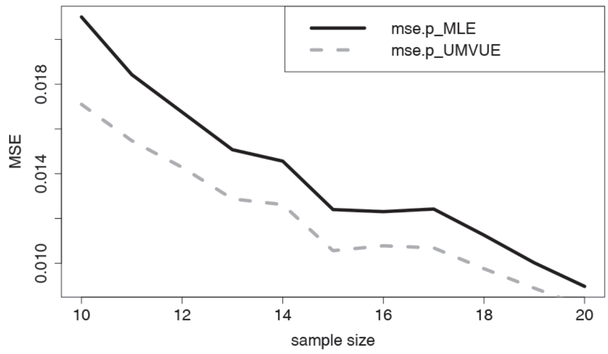
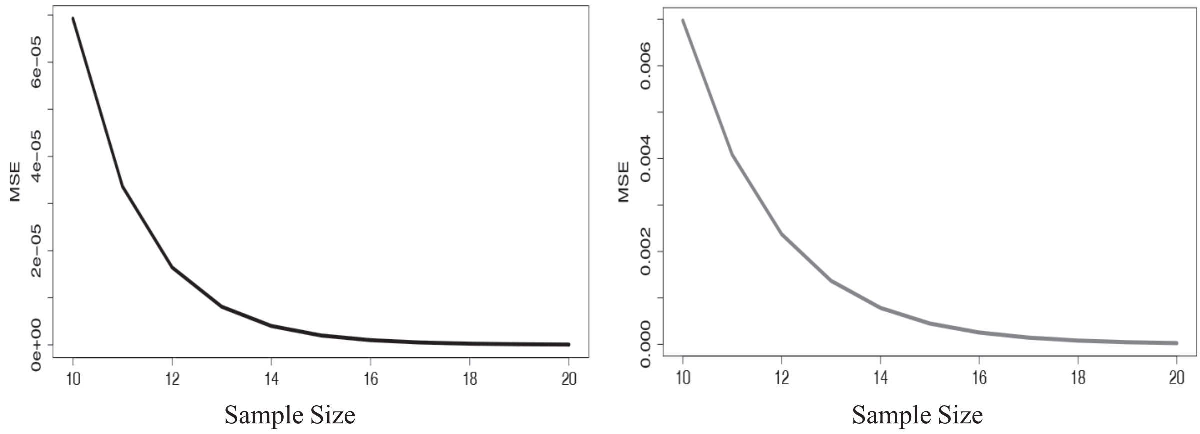


Fig. 1: Graph of simulated MSE of MLE and UMVUE of p



(a) Variance of UMVUE of PMF

(b) Variance of UMVUE of CDF

Fig. 2: Graph of variance of UMVU estimator of the PMF and the CDF for *p* = 0.76 and *x* = 2.

4. PUMVUE of the PMF and the CDF

In this section, we obtain the PUMVUE of the PMF and the CDF of the logarithmic series distribution. Also, we obtain the MSEs of these estimators. The UMVUE of p is given by

$$\widehat{\widehat{p}} = \frac{t|s(t-1,n)|}{|s(t,n)|}; t > n \tag{8}$$

where, $t = \sum_{i=1}^{n} x_i$ [14]. Therefore, we obtain the PUMVUE of the PMF and the CDF as

$$\widehat{\widehat{f}}(x) = \frac{-1}{\ln(1-\widehat{\widehat{p}})} \cdot \frac{\widehat{\widehat{p}}^{x}}{x} \tag{9}$$

$$\widehat{\widehat{F}}(x) = \sum_{w=1}^{x} \frac{-1}{\ln(1-\widehat{\widehat{p}})} \cdot \frac{\widehat{\widehat{p}}^{w}}{w} \tag{10}$$

The bias of PUMVUE of the PMF is

$$B(\widehat{\widehat{f}}(x)) = \sum_{t=n+1}^{\infty} \widehat{\widehat{f}}(x) \cdot g(t;n,p) - f(x). \tag{11}$$

By substituting (1), (4), (8) and (9), we get the value of (11).

The bias of PUMVUE of the CDF is

$$B(\widehat{\widehat{F}}(x)) = \sum_{t=n+1}^{\infty} \widehat{\widehat{F}}(x) \cdot g(t; n, p) - F(x) \tag{12}$$

Using (2), (4), (8) and (10), we get the value of (12).

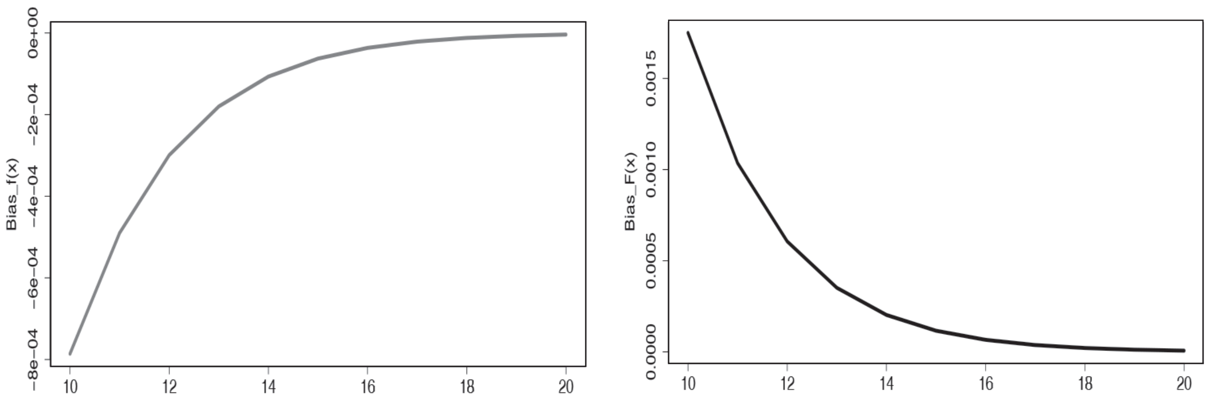
The MSE of PUMVUE of the PMF is obtained by

$$MSE(\widehat{\widehat{f}}(x)) = \sum_{t=n+1}^{\infty} (\widehat{\widehat{f}}(x) - f(x))^2 g(t; n, p). \tag{13}$$

We get the value of (13) using (1), (4), (8) and (9). Similarly,

$$MSE(\widehat{\widehat{F}}(x)) = \sum_{t=n+1}^{\infty} (\widehat{\widehat{F}}(x) - F(x))^2 g(t; n, p). \tag{14}$$

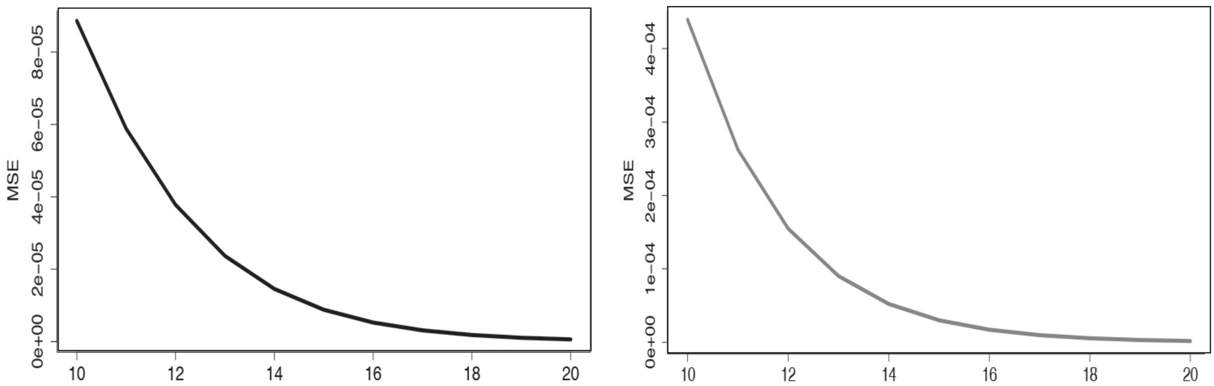
We get the value of (14) using (2), (4), (8) and (10). Though we are not able to have any simplified form, we get the value of bias and MSE using R software. Bias and MSE of PUMVUE of the PMF and the CDF for different sample sizes are presented in Figures 3 and 4 respectively.



Sample Size
(a) Bias of PUMVUE of PMF

Sample Size
(b) Bias of PUMVUE of CDF

Fig. 3: Graph of bias of PUMVUE of the PMF and the CDF for *p* = 0.76 and *x* = 2.



Sample Size
(a) MSE of PUMVUE of PMF

Sample Size
(b) MSE of PUMVUE of CDF

Fig. 4: Graph of variance of PUMVUE of the PMF and the CDF for *p* = 0.76 and *x* = 2.

5. Least squares and weighted least squares estimators

The least squares estimator (LSE) and weighted least squares estimator (WLSE) were proposed by [15] to estimate the parameters of Beta distribution. In this paper, we apply the same technique for the logarithmic series distribution. Suppose $X_1, \ldots, X_n$ is a random sample of size n from a CDF $F(.)$ and let $X_{i:n}$, $i = 1, \ldots\ n$ denote the ordered sample in ascending order. The proposed method uses the CDF, $F(X_{i:n})$. For a sample of size n, we have $E[F(X_{j:n})] = \frac{j}{n+1}$, $Var[F(X_{j:n})] = \frac{j(n-j+1)}{(n+1)^2(n+2)}$ and $Cov[F(X_{j:n}); F(X_{k:n})] = \frac{j(n-k+1)}{(n+1)^2(n+2)}$ for j < k, [see [16]]. Using the expectations and the variances, two variants of the least squares method follow.

Method 1: Least squares estimator

This method is based on minimizing

$\sum_{j=1}^{n}[F(X_{j:n}) - \frac{j}{n+1}]^2$

with respect to the unknown parameters.

In case of logarithmic series distribution the LSE of p is $\tilde{p}_{LSE}$, which can be obtained by minimizing

$\sum_{j=1}^{n}[\sum_{w=1}^{x(j:n)} \frac{-1}{\ln(1-p)}\frac{p^w}{w} - \frac{j}{n+1}]^2$ with respect to p, where $x_{j:n}$ is the observed value of $X_{j:n}$.

So, to obtain the LSE of the PMF and the CDF, we use the same method as is used for the MLE. Therefore, the LSE of the PMF, $\hat{f}_{LSE}(x)$ and the CDF, $\hat{f}_{LSE}(x)$ are given by

$$\tilde{f}_{LSE}(x) = \frac{-1}{\ln(1-\tilde{p}_{LSE})}\frac{\tilde{p}_{LSE}^x}{x} \tag{15}$$

and

$$\tilde{F}_{LSE}(x) = \sum_{w=1}^{x}\frac{-1}{\ln(1-\tilde{p}_{LSE})}\frac{\tilde{p}_{LSE}^w}{w}, \tag{16}$$

respectively. It is difficult to find the expectation and the MSE of these estimators analytically, so we calculate them by means of simulation study.

Method 2 : Weighted Least squares estimator

This method is based on minimizing

$\sum_{j=1}^{n} w_j[F(X_{j:n}) - \frac{j}{n+1}]^2$

with respect to the unknown parameters, where

$w_j = \frac{1}{Var[F(X_{j:n})]} = \frac{(n+1)^2(n+2)}{j(n-j+1)}$

In case of the logarithmic series distribution, the WLSE of p, say, $\tilde{p}w_{LSE}$ is the value minimizing

$\sum_{j=1}^{n} w_j[\sum_{w=1}^{x(j:n)} \frac{-1}{\ln(1-p)}\frac{p^w}{w} - \frac{j}{n+1}]^2$.

So, the WLSE of the PMF and the CDF are

$$\tilde{f}_{WLSE}(x) = \frac{-1}{\ln(1-\tilde{p}_{WLSE})}\frac{\tilde{p}_{WLSE}^x}{x} \tag{17}$$

and

$$\tilde{F}_{WLSE}(x) = \sum_{w=1}^{x}\frac{-1}{\ln(1-\tilde{p}_{WLSE})}\frac{\tilde{p}_{WLSE}^w}{w} \tag{18}$$

The average and the MSE of these estimators have been calculated by means of a simulation study.

6. Estimation of Reliability function and some related measures

It is to be noted that we will have the MLE and the UMVUE of $R(x)$, the reliability function but for other related measures they will be plug-in estimators.

1. Reliability function :

$$\begin{aligned} R(x) &= P(X \geq x) \\ &= 1 - F(x-1) \\ &= 1 - \sum_{w=1}^{x-1} \frac{-1}{\ln(1-p)} \frac{p^w}{w} \end{aligned} \tag{19}$$

2. Hazard rate function :

$$\begin{aligned} h(x) &= \frac{f(x)}{R(x)} \\ &= \frac{p^x}{x} \left[-\ln(1-p) - \sum_{w=1}^{x-1} \frac{p^w}{w} \right]^{-1} \end{aligned} \tag{20}$$

3. Hazard rate average function :

$$\begin{aligned} A(x,p) &= \frac{1}{x} \sum_{i=1}^{x} h(i) \\ &= \frac{1}{x} \sum_{i=1}^{x} \frac{p^i}{i} \left[-\ln(1-p) - \sum_{w=1}^{i-1} \frac{p^w}{w} \right]^{-1} \end{aligned} \tag{21}$$

4. Aging intensity function :

$$\begin{aligned} L(x,p) &= \frac{h(x)}{A(x,p)} \\ &= p^x \left[-\ln(1-p) - \sum_{w=1}^{x-1} \frac{p^w}{w} \right]^{-1} \\ &\quad \times \frac{1}{\sum_{i=1}^{x} \frac{p^i}{i} \left[-\ln(1-p) - \sum_{w=1}^{i-1} \frac{p^w}{w} \right]^{-1}} \end{aligned} \tag{22}$$

5. Mean remaining life function :

$$\begin{aligned} m(x) &= \frac{1}{R(x)} \sum_{i=x}^{\infty} R(i) \\ &= \frac{\sum_{i=x}^{\infty} \left[1 - \sum_{w=1}^{i-1} \frac{-1}{\ln(1-p)} \frac{p^w}{w} \right]}{1 - \sum_{w=1}^{x-1} \frac{-1}{\ln(1-p)} \frac{p^w}{w}} \end{aligned} \tag{23}$$

We have the MLE, $\tilde{p}$ of p from (3) and hence by substituting $\tilde{p}$ in the expressions (19)-(23), we have the corresponding MLEs. Again, we have the UMVUE of the PMF, $\hat{f}(x)$ and that of the CDF, $\hat{F}(x)$ in (6) and (7). By substituting these in (19)-(23), we have the PUMVUEs of the corresponding functions.

In Figures 6-8 the estimated reliability function, hazard rate function, hazard rate average function, aging intensity function, mean remaining life function are presented for the data set [17].

7. Simulation study

Here, we conduct Monte Carlo simulation to evaluate the performance of the estimators for the PMF and the CDF discussed in the previous sections. All computations were performed using the R-software. We evaluate the performance of the estimators based on MSEs. The MSEs were computed by generating 1000 replications from logarithmic series distribution for p = 0.76 and $x = 2$. It is observed from Figure 5 that MSEs decrease with increasing sample size. It veries the consistency properties of all the estimators. We observe from true MSE point of view, MLE is better than other estimators for both the PMF and the CDF.

We have discussed here sample generation procedure [see, [18]] from the logarithmic series distribution.

1. Set $t = \frac{-p}{\ln(1-p)}$.
2. Generate U from *uniform* (0, 1) and set $x = 1$; $\alpha = t$.
3. If $U \leq \alpha$, deliver x as the generated value.
4. Otherwise replace U by $U - \alpha$, x by $x + 1$, α by $\frac{\alpha p(x-1)}{x}$ and then go to step 3.

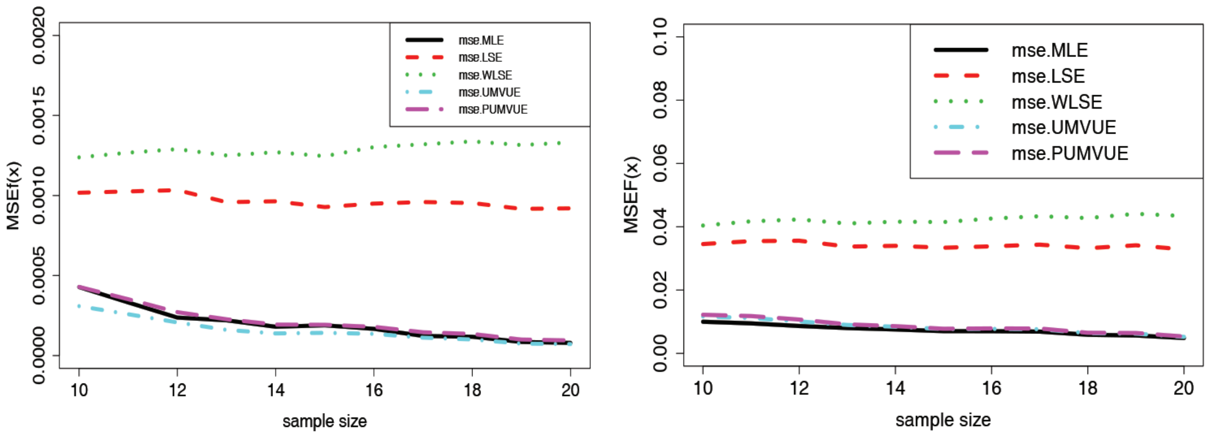


Fig. 5: MSEs of LSE, WLSE, MLE, PUMVUE and UMVUE of the PMF and the CDF of the logarithmic series distribution for p = 0.76 and x = 2.

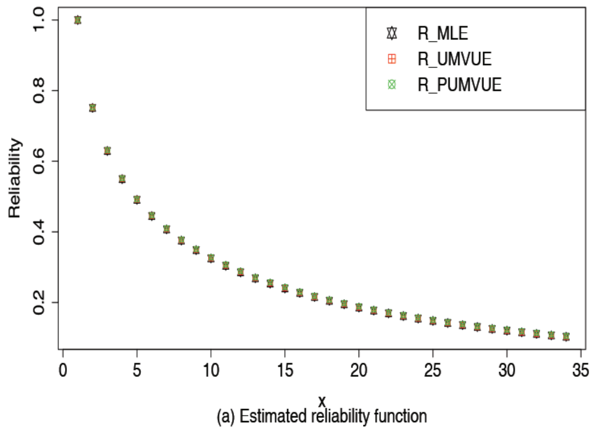
8. Application

In this section, we provide the analysis of real data set for illustrative purpose. We have compared the performances of MLE, PUMVUE, UMVUE, LSE, WLSE of the PMF and the CDF for the logarithmic series distribution. The graphical presentation of the estimated PMF and CDF of data set is shown in Figure 9.

Here we have studied the data on the number of Macrolepidoptera caught in a light trap in the year of 1934 (given in Table 1). This data set is obtained from [17]. They have reported chi-square goodness-of-fit test having observed $x^2 = 9.16$ with d.f. 14 and the corresponding p-value is 0.82.

Table 1: Number of Macrolepidoptera caught in a light trap in the year of 1934.

Moths per species	1	2	3	4	5	6	7	8	9	10
Number of species in 1934	34	19	15	10	10	6	3	9	5	3
Moths per species	12	13	14	15	16	17	18	19	20	21
Number of species in 1934	1	5	3	6	3	1	1	1	2	2
Moths per species	22	23	24	25	28	29	32	33	38	39
Number of species in 1934	3	2	1	1	1	1	2	1	1	1
Moths per species	40	41	44	48	65	73	75	82	87	90
Number of species in 1934	3	1	1	2	1	1	1	1	1	1
Moths per species	99	100	109	126	138	145	153	159	165	219
Number of species in 1934	1	1	1	1	1	1	1	1	1	1



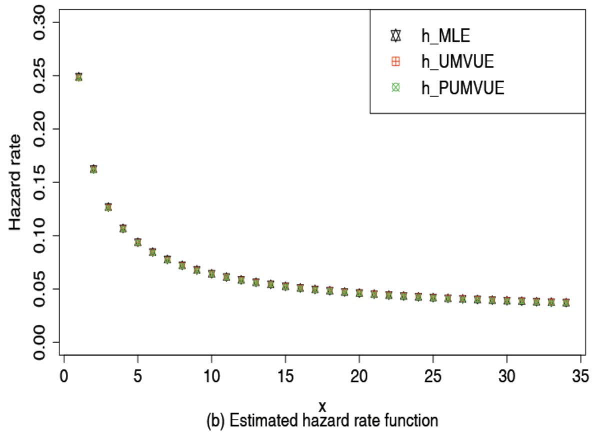
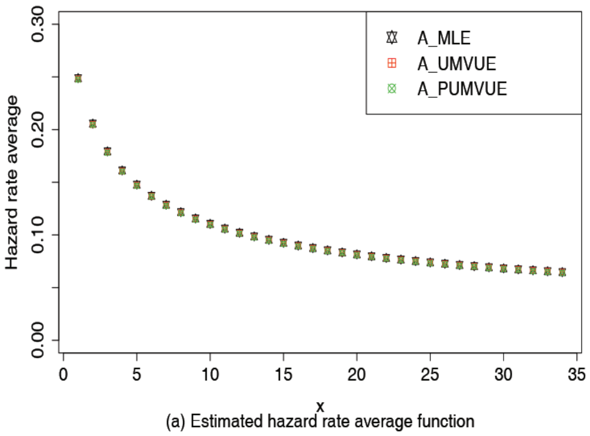


Fig. 6: Graph of MLE, UMVUE and PUMVUE of reliability function and hazard rate function for the data set.



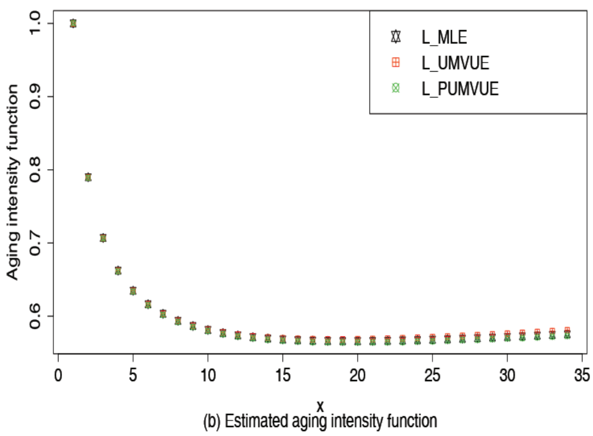


Fig. 7: Graph of MLE, UMVUE and PUMVUE of hazard rate average function and aging intensity function for the data set.

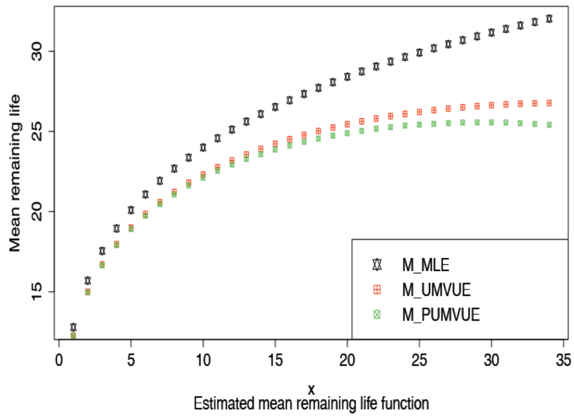


Fig. 8: Graph of MLE, UMVUE and PUMVUE of mean remaining life function for the data set.

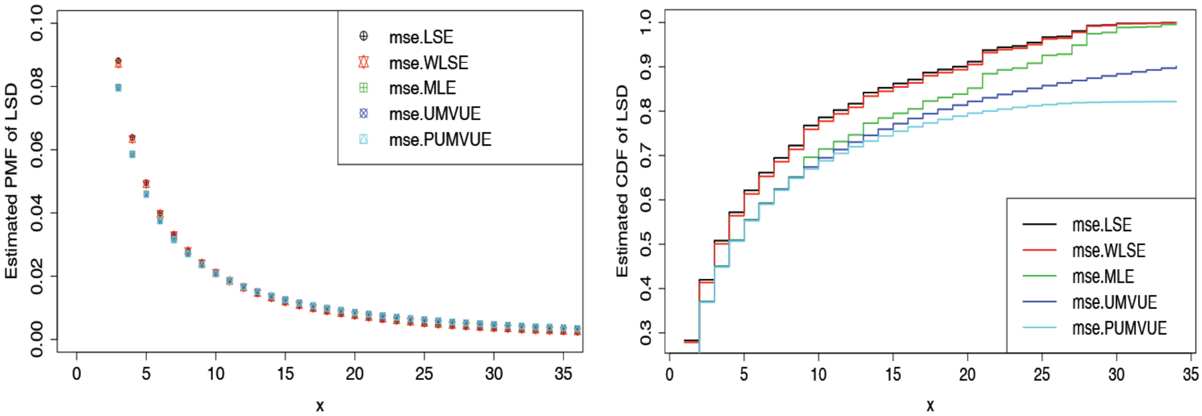


Fig. 9: Estimated PMF and CDF of the logarithmic series distribution.

Table 2: Summarization of negative log-likelihood value

Estimator of PMF	Negative log-likelihood value
WLSE	631.8942
LSE	631.8891
MLE	631.8801
UMVUE	631.859
PUMVUE	631.884

Table 2 gives the estimate of the negative log-likelihood values. Lower the value of negative log-likelihood indicates the better fit. The results are more or less in same tune with the theoretical counterpart. Here UMVUE is better in negative log-likelihood sense.

9. CONCLUSION

In the article, different methods of estimation of the PMF and the CDF of the logarithmic series distribution have been considered. Maximum likelihood estimator (MLE), uniformly minimum variance unbiased estimator (UMVUE), plug-in uniformly minimum variance unbiased estimator (PUMVUE), least squares estimator (LSE), weighted least squares estimator (WLSE) have been found out. Monte Carlo simulations are performed to compare the performances of the proposed estimators. Though UMVUE is better than the MLE of the parameter p, the plug-in UMVUE is not better than the MLE or UMVUE of the PMF and the CDF in MSE sense. Actually UMVUE of the PMF is better, and MLE as well as UMVUE are more or less equally efficient for CDF. It is observed from the data analysis that UMVUE is better than other estimators for negative log-likelihood sense. Estimators of some reliability-related measures have been discussed. Results of analysis of a data set have been reported.

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