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Robustness of BIB Designs for Multi-Response Experiments Against The Loss of Observations

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ABSTRACT

Robustness of Balanced Incomplete Block (BIB) designs for multi-response experiments against loss of observations has been investigated. Two cases of possible pattern of occurrence of missing observations have been considered. In the first case, all observations from a particular plot are lost. In the second case it has been considered that all observations from any two plots are missing. Since in multi-response experiments, response variables are correlated among themselves, some pattern of correlation structures have been considered while studying robustness property. For studying robustness, connectedness criterion and A-efficiency criterion have been considered. A-efficiencies of the BIB designs that are robust as per connectedness criterion have been obtained.

Keywords : A-efficiency, BIB design, connectedness, correlation, multi-response experiment, robustness

1. Introduction

Loss of observations in a designed experiment is a common phenomenon. When one or more observations are lost, the residual design may become disconnected where it would not be possible to make all possible paired comparisons of treatment effects through the design. An experimenter may, therefore, prefer a design that remains connected even after the loss of some observations. Even if the residual design remains connected, the efficiency loss in terms of the variance of treatment contrasts may be very high leading to an inefficient experiment. The experimenter may, therefore, prefer a connected design if the efficiency loss of the residual design, as compared to the original design, is not too high. Several authors have studied this loss of information in terms of A-efficiency criterion, *i.e.*, average of the variances of all treatment contrasts in the designs. Therefore, it is of interest to examine the loss of information that is incurred due to loss of observations. A design that is capable of absorbing such shocks is termed as robust design. Some references in this regard in block designs are Das and Kageyama (1992),Dey (1993), Dey *et al*. (1996), Duan and Kageyama (1995), Lal *et al*. (2001), Mukerjee and Kageyama (1990) and so on.

In many experimental situations, data on more than one response variable is recorded from the same experimental unit through application of the same treatment. Such experiments are known as multi-response experiments. So, a huge number of data are generated in multi-response experiment. Therefore, it is very much possible to have some missing observations due to some unforeseen causes. It is, therefore, important to study the robustness of block design for multi-response experiments as per connectedness criterion as well as per efficiency criterion. In the present investigation an attempt has been made to study these robustness properties of Balanced Incomplete Block (BIB) designs for multi-response experiments. Since in multi-response experiments response variables are correlated among themselves, some particular correlation structures are considered while studying the robustness properties. Firstly, general expressions for robustness have been obtained for the loss of any t observations. This is obtained in section 2. In section 2.1, condition for robustness as per connectedness criterion and in section 2.2 robustness condition as per efficiency criterion has been obtained. In section 3, various correlation structures in multi-response experiments are discussed. Some particular pattern of occurrence of missing observations in BIB designs has been considered in section 4. Conditions for robustness as per connectedness criterion are obtained in terms of the smallest eigenvalue of the C-matrix of the designs under various correlation structures. Expressions for obtaining the efficiency factors are also obtained. These conditions are applied to 494 BIB designs listed in the website of Indian Agricultural Statistics Research Institute, New Delhi (*http:www.iasri.res.in/design*). We termed a design robust as per efficiency criterion, if the efficiency factor is found to be greater than or equal to 0.80.

Throughout the paper we deal with only real matrices and vectors. Denote an n-component vector of all unities by $\mathbf{1}_{n}$-, an identity matrix of order n by $\mathbf{I}_n$ and $m \times n$ matrix of all ones by $\mathbf{J}_{m\times n}$, $\mathbf{J}_{m\times m}$ is simply denoted by $\mathbf{J}_m$. Further, $\mathbf{A}'$, $\mathbf{A}^-$ and $\mathbf{A}^+$ will respectively denote the transpose, a g-inverse and the Moore-Penrose inverse of a matrix **A.**

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2. Condition for robustness

Consider a Balanced Incomplete Block (BIB) design (*d*, say) with usual parameters *v, b, k, r* and λ in which from each experimental unit *p* responses are observed. Let the linear model for such design be

$$y = X\theta + \varepsilon, \tag{1}$$

where $y = \left(y_1^{*\prime}\ y_2^{*\prime} \ldots y_p^{*\prime}\right)'$ is an $np \times 1$ vector of observations, $X = (I_p \otimes X^*)$ is an $np \times mp$ design matrix, $\theta = (\theta_1^{*\prime}\ \theta_2^{*\prime} \ldots \theta_p^{*\prime})'$ is an $mp \times 1$ vector of parameters, $\varepsilon = (\varepsilon_1^{*\prime}\ \ \varepsilon_2^{*\prime}\ \ \ldots\ \ \varepsilon_p^{*\prime})'$ is an $np \times 1$ vector of random errors with y_s^*, X^*, θ_s^* and ε_s^* being their corresponding components for the s^{th} response, $s = 1, 2, \ldots, p$ and $\otimes$ denotes the Kronecker product of matrices. That is, for each response, there are *m* parameters. These *m* parameters is composed of *v* treatment effects, *b* block effects and general mean, *i.e.*, $m = v + b + 1$. Accordingly X^* is partitioned and θ^*_s is composed of general mean, *v* treatments effects and *b* block effects for each $s = 1, 2, \ldots. p$. We also assume that $\varepsilon \sim N_p(0, \Omega)$, *i.e.*, $E(\varepsilon) = 0$ and $D(\varepsilon) = \Omega = \Sigma_{pp} \otimes I_n$, where E and D stand for expected value and dispersion matrix respectively and

$$\Sigma_{pp} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix}.$$

The dispersion matrix Ω is positive definite and symmetric. Thus it is assumed that the observations corresponding to a particular response are uncorrelated and have constant variances, whereas the observations corresponding to a particular experimental unit are correlated among themselves.

We partitioned θ as $\theta = \left[\theta_1'\ \theta_2'\right]'$, where θ_1 is a $vp \times 1$ vector parameters of interest and θ_2 is a $(m-v)p \times 1$ vector of nuisance parameters. Also $X = [X_1\ X_2]$ is partitioned in conformity with the parameters of θ, *i.e.*, X_1 is a $np \times vp$ design matrix for the parameters of interest (say treatment effects) and X_2 is a $np \times (m - v)\,p$ design matrix for the nuisance parameters. The usual **C**-matrix for obtaining the best linear unbiased estimators (BLUE) of linear parametric functions of θ_1, after eliminating nuisance parameters θ_2 for the multi-response experiment of order *vp* can be obtained in terms of the corresponding **C**-matrix of the uni-response experiment as

$$C_{\theta_1} = \Sigma_{pp}^{-1} \otimes C^*_{\theta_1^*}$$

where $C^*_{\theta_1^*}$ is the **C**-matrix in the usual set up for the uni-response case. A design is said to be connected for parameters θ_1, iff the rank of C_{θ_1} is $p(v-1)$. We assume that the design (*d*) is connected and rank of C_{θ_1} is $p(v-1)$.

Suppose now that any $t\ (\geq 1)$ observations are lost and we consider the model with *t* missing observations as

$$M_{(t)} = \left(y_{(t)}, X_{(t)}\theta, \Sigma_{pp} \otimes I_{(t)}\right) \tag{2}$$

where $\mathbf{y}_{(t)}$ have $p(n\text{-}t)$ observations, $\mathbf{I}_{(t)}$ and $\mathbf{X}_{(t)}$ has $(n\text{-}t)$ and $p(n\text{-}t)$ rows respectively. We denote the residual design as $d_{(t)}$. The **C**- matrix under this model $M_{(t)}$ can easily be obtained as

$$C_{\theta_{1(t)}} = \Sigma_{pp}^{-1} \otimes C^*_{\theta^*_{1(t)}}. \tag{3}$$

As a third model, we consider the following linear model in which we devote an extra parameter to each missing observation

$$M_{(Z)} = \left\{y, X\theta_1 + X_2\theta_2 + U\delta, A\Omega A'\right\}, \tag{4}$$

where $U_{np \times tp} = \left(I_p \otimes U^*_{n \times t}\right)$ is a $np \times tp$ matrix and $U^*_{n \times t} = \left[u_1\ \ u_2\ \ \ldots\ \ u_t\right]$ is a $n \times t$ matrix having one element as 1(if j^{th} observation is missing) and rest are zero for each column, *i.e.*,
$u_i' = \left(0\ \ \ldots\ \ 1(j^{th})\ \ 0\ \ \ldots\ \ 0\right)$,
$\forall\ j = 1, 2, \ldots, n\ \&\ i = 1, 2, \ldots, t$. δ is a $tp \times 1$ vector of new parameters added due to *t* missing observations for each response. We denote the usual **C**-matrix under $M_{(Z)}$ by $\mathbf{C}_{\theta_{1(Z)}}$. Bhar and Ojha (2012) shown that the **C**-matrix under $M_{(Z)}$ in (4), *i.e.*, $\mathbf{C}_{\theta_{1(Z)}}$ is identical to

the **C**-matrix under $M_{(t)}$ in (2), *i.e.*, $\mathbf{C}_{\theta_{1(t)}}$. The matrix $\mathbf{C}_{\theta_{1(Z)}}$ can be expressed as (see Bhar and Gupta, 2002):

Lemma 2.1:

$$C_{\theta_{1(Z)}} = C_{\theta_1} - V'C_0^- V \tag{5}$$

Where $V = X_1BU.$, $C_0 = U'BU$, $B = I_p \otimes B^*$ and $B^* = (I - X_2^*(X_2^{*\prime}X_2^*)^- X_2^{*\prime})$.

After substituting

$$X_1 = I_p \otimes X_1^*,\ X_2 = I_p \otimes X_2^*, \Omega^{-1} = \Sigma_{pp}^{-1} \otimes I_n$$

and $U = (I_p \otimes U^*)$ and simplifying, we get

$$V'C_0^- V = \sum_{pp}^{-1} \otimes V^{*\prime}C_0^{*-}V^* \tag{6}$$

The matrices V^*, C_0^* and U^* are corresponding matrices in uni-response experiment.

2.1 Robustness as per connectedness criterion

We now give the necessary and sufficient condition for a connected design *d*, under a general linear model set-up, to be robust against the loss of any t (≥ 1) observations.

Theorem 2.1:

The design $d_{(t)}$ is robust as per connectedness criterion against the loss of any t (≥ 1) observations if and only if

$$\mathbf{I}_p - \mathbf{C}_0^{-\frac{1}{2}}\mathbf{V}'\mathbf{C}_{\theta_1}^-\mathbf{V}\mathbf{C}_0^{-\frac{1}{2}} \text{ is positive definite.}$$

Proof: The proof follows on the lines of proof of Theorem 1 and Theorem 3 of Dey (1993) and noting that $\mathbf{C}_0^-$ admits a unique Gramian root.

Remark 2.1: From Theorem 2.1, the design $d_{(t)}$ is robust as per connectedness criterion against the loss of any t observations if and only if $\lambda_{max}\left(\mathbf{C}_0^{-\frac{1}{2}}\mathbf{V}'\mathbf{C}_{\theta_1}^-\mathbf{V}\mathbf{C}_0^{-\frac{1}{2}}\right) < 1$, where $\lambda_{\max}(\mathbf{A})$ is the largest eigenvalue of A, *i.e.*, the smallest positive eigenvalue of $\mathbf{C}_{\theta_1}$ is strictly greater than the largest eigenvalue of $\mathbf{V}\mathbf{C}_0^-\mathbf{V}'$.

Now, $\lambda_{max}\left(\mathbf{C}_0^{-\frac{1}{2}}\mathbf{V}'\mathbf{C}_{\theta_1}^-\mathbf{V}\mathbf{C}_0^{-\frac{1}{2}}\right) =$

$$\lambda_{max}\left(\mathbf{C}_{\theta_1}^+\mathbf{V}\mathbf{C}_0^-\mathbf{V}'\right) = \lambda_{max}\left(\mathbf{C}_{\theta_1}^+\mathbf{V}\mathbf{C}_0^-\mathbf{V}'\right) \tag{7}$$

It is known that (Marshall and Olkin, 1979) for a pair of symmetric, nonnegative definite matrices **A** and **B,**

$$\lambda_{max}(\mathbf{AB}) \leq \lambda_{max}(\mathbf{A})\lambda_{max}(\mathbf{B}) \tag{8}$$

Hence from (7) and (8), we have

$$\lambda_{max}\left(\mathbf{C}_{\theta_1}^+\mathbf{V}\mathbf{C}_0^-\mathbf{V}'\right) \leq \lambda_{max}\left(\mathbf{C}_{\theta_1}^+\right)\lambda_{max}\left(\mathbf{V}\mathbf{C}_0^-\mathbf{V}'\right) \tag{9}$$

Therefore the design $d_{(t)}$ is robust as per connectedness criterion against the loss of any t observations if and only

$\lambda_{max}\left(\mathbf{C}_{\theta_1}^+\right)\lambda_{max}\left(\mathbf{V}\mathbf{C}_0^-\mathbf{V}'\right) < 1$ and remembering that $\lambda_{max}\left(\mathbf{C}_{\theta_1}^+\right) = \left\{\lambda_{min}\left(\mathbf{C}_{\theta_1}\right)\right\}^{-1}$, where $\lambda_{min}(\mathbf{A})$ is the smallest eigenvalue of **A,** we get the condition as $\lambda_{max}\left(\mathbf{V}\mathbf{C}_0^-\mathbf{V}'\right) < \lambda_{min}\left(\mathbf{C}_{\theta_1}\right)$,

Now using the values of $\mathbf{C}_{\theta_1}$ and **V,** we get

$$\lambda_{max}\left(\Sigma^{-1}\right)\lambda_{max}\left(\mathbf{V}^*\mathbf{C}_0^{*-}\mathbf{V}^{*\prime}\right) < \lambda_{min}\left(\Sigma^{-1}\right)\lambda_{min}\left(\mathbf{C}_{\theta_1}^*\right)$$

$$\Rightarrow \lambda_{min}\left(\mathbf{C}_{\theta_1}^*\right) > \frac{\lambda_{max}(\Sigma)}{\lambda_{min}(\Sigma)}\lambda_{max}\left(\mathbf{V}^*\mathbf{C}_0^{*-}\mathbf{V}^{*\prime}\right) \tag{10}$$

For a BIB design, $\mathbf{C}_{\theta_1}^* = \theta\left(\mathbf{I}_v - \frac{1}{v}\mathbf{J}_v\right)$ where θ is the unique positive eigenvalue of $\mathbf{C}_{\theta_1}^*$, thus condition of robustness as per connectedness criterion becomes

$$\theta > \frac{\lambda_{max}(\Sigma)}{\lambda_{min}(\Sigma)}\lambda_{max}\left(\mathbf{V}^*\mathbf{C}_0^{*-}\mathbf{V}^{*\prime}\right) \tag{11}$$

2.3 Robustness as per efficiency criterion

Once it is known that a design is robust as per connectedness criterion, then it is of interest to examine the efficiency of residual design relative to original design and to decide robustness on the basis of efficiency. The efficiency of the residual design with respect to the original design is given by

$$E = \frac{\text{Sum of reciprocals of non-zero eigenvalues of } \mathbf{C}_{\theta_1}}{\text{Sum of reciprocals of non-zero eigenvalues of } \mathbf{C}_{\theta_{1(Z)}}} = \frac{\text{trace}\left(\mathbf{C}_{\theta_1}^+\right)}{\text{trace}\left(\mathbf{C}_{\theta_{1(Z)}}^+\right)}$$

where $\mathbf{C}_{\theta_1}$ and $\mathbf{C}_{\theta_{1(Z)}}$ are the **C**-matrices of d and $d_{(t)}$ respectively. Now on using Theorem 2 of Dey (1993) from (5), we get

$$\mathbf{C}^{+}_{\theta_{1(Z)}} = \mathbf{C}^{+}_{\theta_1} + \mathbf{C}^{+}_{\theta_1}\mathbf{V}\mathbf{C}_0^{-\frac{1}{2}}\left(\mathbf{I}_{np} - \mathbf{C}_0^{-\frac{1}{2}}\mathbf{V}'\mathbf{C}^{+}_{\theta_1}\mathbf{V}\mathbf{C}_0^{-\frac{1}{2}}\right)^{-1}\mathbf{C}_0^{-\frac{1}{2}}\mathbf{V}'\mathbf{C}^{+}_{\theta_1}$$

Thus,

$$\text{trace}\left(\mathbf{C}^{+}_{\theta_{1(Z)}}\right) = \text{trace}\left(\mathbf{C}^{+}_{\theta_1}\right)$$

$$+\ \text{trace}\left[\mathbf{C}^{+}_{\theta_1}\mathbf{V}\mathbf{C}_0^{-\frac{1}{2}}\left(\mathbf{I}_{np} - \mathbf{C}_0^{-\frac{1}{2}}\mathbf{V}'\mathbf{C}^{+}_{\theta_1}\mathbf{V}\mathbf{C}_0^{-\frac{1}{2}}\right)^{-1}\mathbf{C}_0^{-\frac{1}{2}}\mathbf{V}'\mathbf{C}^{+}_{\theta_1}\right]$$

$$= \text{trace}\left(\mathbf{C}^{+}_{\theta_1}\right) + \delta$$

where,

$$\delta = \text{trace}\left[\mathbf{C}^{+}_{\theta_1}\mathbf{V}\mathbf{C}_0^{-\frac{1}{2}}\left(\mathbf{I}_{np} - \mathbf{C}_0^{-\frac{1}{2}}\mathbf{V}'\mathbf{C}^{+}_{\theta_1}\mathbf{V}\mathbf{C}_0^{-\frac{1}{2}}\right)^{-1}\mathbf{C}_0^{-\frac{1}{2}}\mathbf{V}'\mathbf{C}^{+}_{\theta_1}\right]$$

$$= \text{trace}\left[\left(\mathbf{I}_{np} - \mathbf{C}^{+}_{\theta_1}\mathbf{V}\mathbf{C}_0^{-}\mathbf{V}'\right)^{-1}\mathbf{C}^{+}_{\theta_1}\mathbf{C}^{+}_{\theta_1}\mathbf{V}\mathbf{C}_0^{-}\mathbf{V}'\right] \tag{12}$$

Thus,

$$E = \frac{\text{trace}\left(\mathbf{C}^{+}_{\theta_1}\right)}{\text{trace}\left(\mathbf{C}^{+}_{\theta_1}\right) + \delta} = \left[1 + \frac{\delta}{\text{trace}\left(\mathbf{C}^{+}_{\theta_1}\right)}\right]^{-1} \tag{13}$$

This expression involves the matrix Σ. Therefore, the eigenvalues of Σ are required for calculation of E. The eigenvalues of Σ depend on the actual data. We, therefore, obtained a lower bound of this efficiency factor.

Now,

$$\mathrm{I}_{np} - \mathrm{C}^{+}_{\theta_1}\mathrm{V}\mathrm{C}_0^{-}\mathrm{V}' = \mathrm{I}_{np} - \left(\Sigma \otimes \mathrm{C}^{*+}_{\theta_1}\right)\left(\Sigma^{-1} \otimes \mathrm{V}^{*}\mathrm{C}_0^{*-}\mathrm{V}^{*\prime}\right) =$$

$$\mathrm{I}_{np} - \left(\mathrm{I}_p \otimes \mathrm{C}^{*+}_{\theta_1}\mathrm{V}^{*}\mathrm{C}_0^{*-}\mathrm{V}^{*\prime}\right)$$

and

$$\mathrm{C}^{+}_{\theta_1}\mathrm{C}^{+}_{\theta_1}\mathrm{V}\mathrm{C}_0^{-}\mathrm{V}' = \Sigma \otimes \mathrm{C}^{+}_{\theta_1}\mathrm{C}^{+}_{\theta_1}\mathrm{V}^{*}\mathrm{C}_0^{*-}\mathrm{V}^{*\prime}$$

Now let, $0 > e_1 \geq e_2 \geq \ldots\ldots \geq e_m$ be the m ($\leq t$) positive eigenvalues of an $n \times n$ matrix and $\mathrm{V}^{*}\mathrm{C}_0^{*-}\mathrm{V}^{*\prime}$ and $0 > l_1 \geq l_2 \geq \ldots\ldots \geq l_p$ be the p positive eigenvalues of Σ_{pp}. Then δ can be written as

$$\delta = \sum_{i=1}^{p}\sum_{j=1}^{m}\frac{p e_j l_i}{\theta\left(\theta - e_j\right)} \tag{14}$$

Now,

$$\text{trace}\left(\mathbf{C}^{+}_{\theta_1}\right) = \text{trace}\left(\Sigma \otimes \mathbf{C}^{*+}_{\theta_1}\right) = \frac{v-1}{\theta}\sum_{i=1}^{p} l_i = \frac{v-1}{\theta} q \text{ (say)},$$

where $q = \sum_{i=1}^{p} l_i$ and hence a lower bound of the efficiency factor E is given by

$$E \geq \left[1 + \sum_{j=1}^{m} p e_j \frac{max(l_i)}{(v-1)\left(\theta - e_j\right)q}\right]^{-1} \tag{15}$$

3 Correlation Structures of Σ

From (11) it is evident that the robustness of designs depends on the eigenvalues of Σ and the eigenvalues of Σ depend on the structure of Σ. In multi-response experiments, where observations are taken on a number of characters from a particular plot, it is very likely that the response variables are correlated among themselves. The pattern of this correlation structure depends on a particular experiment conducted and types of characters on which observations are taken. However, we may expect equal correlations among these response variables when the response variables are similar in nature. Another situation may also occur where we can arrange the response variables in such a manner so that the correlation coefficient will be high for the nearest two variables and will become lesser as soon as we move further away from the first variable, i.e., a correlation structure we observed in time series analysis. These two correlation structures are described below:

3.1 Correlation Structure A: Equi-Correlation Structure (EC)

In case of the equi-correlation structure, it is assumed that the same amount of correlation (ρ) exists between the response variables. The amount of correlation is constant for all pair of variables. The correlation between (y_j, y_j) is same for all $j \neq j' = 1,\ldots\ldots,p$.

$$\text{Corr}\left(\mathbf{y}_j, \mathbf{y}_{j'}\right) = \begin{cases}\rho & \text{if } \quad j \neq j' \\ 1, & \text{otherwise}\end{cases} \tag{16}$$

In this case the variance-covariance matrix (Σ) can be given as

$$\Sigma = \sigma^2\left[(1-\rho)\mathbf{I} + \rho\mathbf{11}'\right]$$

The eigenvalues of Σ matrix are as follows

$\sigma^2(1-\rho)$ with multiplicity (p-1) and

$\sigma^2\left[1 + (p-1)\rho\right]$ with multiplicity 1. (17)

3.2 Correlation Structure B: A specific pattern

In this correlation structure, it is generally assumed that the same amount of correlation (ρ) exists between the variables in the nearest manner as discussed earlier. Here we assume the correlation decreases at a geometric progression rate among the variables. For this type of correlation structure the correlation between $(\mathbf{y}_j, \mathbf{y}_{j'})$ would be as follows:

$$\text{Corr}\,(\mathbf{y}_j, \mathbf{y}_{j'}) = \begin{cases} \rho^g & \text{if } j \neq j' \text{ and } |j - j'| = g \\ 1, & \text{otherwise} \end{cases} \tag{18}$$

In this case the variance-covariance matrix (Σ) can be given as

$$\Sigma = \sigma^2\left[(1-\rho^g)\mathbf{I} + \rho^g \mathbf{11'}\right]$$

The eigenvalues of Σ matrix are as follows

$\sigma^2(1-\rho^g)$ with multiplicity (p-1) and

$\sigma^2\left[1+(p-1)\rho\right]$ with multiplicity 1. (19)

4 Robustness of BIB design

We now study the robustness of BIB designs as per both the criteria for various autocorrelation structures as described in section 3 for occurrence of pattern of missing observations.

4.1 Robustness when all observations from a plot are lost

Since p observations pertain to a single plot, one observation corresponds to each of the response variables. The matrix $\mathbf{V}^*\mathbf{C}_0^{*-}\mathbf{V}^{*\prime}$ is for a single response in BIB designs. We, therefore, need to calculate the eigenvalues of $\mathbf{V}^*\mathbf{C}_0^{*-}\mathbf{V}^{*\prime}$, *i.e.*, when a single observation is lost from BIB design for uni-response experiment. Without loss of generality we can assume that the missing observation is pertaining to first treatment in the first block. In this case, it can easily be seen that the only positive eigenvalue of $\mathbf{V}^*\mathbf{C}_0^{*-}\mathbf{V}^{*\prime}$ is 1. Thus the condition for connectedness from (11) reduces to

$$\theta > \frac{\lambda_{max}(\Sigma)}{\lambda_{min}(\Sigma)} \tag{20}$$

Thus the design $d_{(t)}$ is robust as per connectedness criterion against the loss of all observations from a plot if and only if (20) is satisfied. For autocorrelation structure A, using the eigenvalues of Σ as given in (17) we get $\frac{\lambda_{max}(\Sigma)}{\lambda_{min}(\Sigma)} = \frac{1+(p-1)\rho}{1-\rho} = 1 + \frac{p\rho}{1-\rho}$. Thus we have the following result:

Theorem 4.1: BIB designs for multi-response experiments with correlation structure A are robust as per connectedness criterion against loss of all observations in a plot, if and only if $\theta > 1 + \frac{p\rho}{1-\rho}$.

Similarly, for correlation structure B, we get

$$\frac{\lambda_{max}(\Sigma)}{\lambda_{min}(\Sigma)} = \frac{1+(p-1)\rho^g}{1-\rho^g} = 1 + \frac{p\rho^g}{1-\rho^g}.$$

Theorem 4.2: BIB designs for multi-response experiments with correlation structure B are robust as per connectedness criterion against loss of all observations in a plot, if and only if $\theta > 1 + \frac{p\rho^g}{1-\rho^g}$.

Remark 4.1: From the expressions in Theorem 4.1 and Theorem 4.2, it is evident that when the correlation coefficient ρ tends to 1, the condition for robustness become $\theta > 1$. It is known that the eigenvalues of **C**-matrix of a BIB design is always greater than 1. Therefore, when the response variables are uncorrelated, a BIB design is always robust as per connectedness criterion. Thus, the situation is like uni-response experiments. However, when $\rho \to 1$, the condition of robust becomes $\theta > p + 1$. Therefore, robustness of BIB design depends on the number of response variables p. We have worked out this condition for 494 BIB designs listed in the website of IASRI, New Delhi (*http:www.iasri.res.in/design*) and enlisted the designs that are robust as per connectedness criterion for different values of p and ρ. The list is not provided here. But it can be checked easily for various values of p and ρ. For an example, for a BIB design with parameters $v = 7$, b = 14, $r = 8$, $k = 4$ and $\lambda = 4$, the design is robust for a value of p up to 4 at $\rho = 0.6$ under correlation structure A, beyond $p = 4$, the design is not robust at this correlation value.

The lower bound of the efficiency factor for correlation structure A from (15) can be obtained as E_A, where

$$E_A \geq \left[1 + \frac{1+(p-1)\rho}{(v-1)(\theta-1)}\right]^{-1}$$

Similarly for correlation structure B, this bound is given by

$$E_B \geq \left[1 + \frac{1+(p-1)\rho^g}{(v-1)(\theta-1)}\right]^{-1}$$

Remark 4.2: From the lower bounds of the efficiency factor, it is evident that for a correlation value near 0, these efficiency factors would be very high and as ρ increases, the value of efficiency factors decrease. Thus again, we see that when the response variables are uncorrelated, the efficiency factor becomes same as we get for a uni-response experiments. For higher value of ρ , the robustness of BIB design depends on p, the number of response variables. We also worked out these efficiency factors under both the correlation structure and enlisted the designs whose efficiency factors are greater than or equal to 0.80 (list is not provided). For example, for a BIB design with parameters $v = 9$, $b = 12$, $r = 4$, $k = 3$, and $\lambda = 1$, the efficiency of this design with $\rho=0.5$ under correlation structure A, is 0.91 for $p \leq 2$.

4.2 Robustness of BIB designs when all observations from any two plots are lost

The loss of any two plots means the loss of two observations corresponding to each of the response variables. Therefore, for studying robustness, we need to calculate the eigenvalues of $\mathbf{V}'^{*}\mathbf{C}_0^{*-1}\mathbf{V}^{*}$ with $\mathbf{U}^{*} = (\mathbf{u}_1\ \mathbf{u}_2)$ for the loss of any two observations in BIB design for uni-response experiment. The eigenvalues of $\mathbf{V}'^{*}\mathbf{C}_0^{*-1}\mathbf{V}^{*}$ have been obtained by Lal *et al*. (2001) for various cases. These eigenvalues along with various cases which may arise in such situation are described below:

Case	Pattern of occurrence	Eigen values
1	Two plots belong to the same block	1 with multiplicity (wm) 2
2(i)	Two plots belong to two different blocks but same treatment	$e_1 = 1 + \frac{k^2 - 2k + \alpha}{k(k-1)}$ and $e_2 = 1 - \frac{k^2 - 2k + \alpha}{k(k-1)}$
2(ii)	Two plots belong to two different blocks but different treatments and two treatments are common in both blocks	$e_1' = 1 + \frac{2k - \alpha}{k(k-1)}$ and $e_2' = 1 - \frac{2k - \alpha}{k(k-1)}$
2(iii)	Two plots belong to two different blocks but different treatments and two treatments are not common in both blocks	$e_1'' = 1 + \frac{\alpha}{k(k-1)}$ and $e_2'' = 1 - \frac{\alpha}{k(k-1)}$

The conditions for robustness for these cases along with the efficiency factors under various correlation structures are described below:

Case	Correlation structure A		Correlation structure B	
	Connectedness	Efficiency	Connectedness	Efficiency
1	$\theta > 1 + \frac{p\rho}{1-\rho}$	$E_{A1} \geq \left[1 + 2\frac{1+(p-1)\rho}{(v-1)(\theta-1)}\right]^{-1}$	$\theta > 1 + \frac{p\rho^g}{1-\rho^g}$	$E_{B1} \geq \left[1 + 2\frac{1+(p-1)\rho^g}{(v-1)(\theta-1)}\right]^{-1}$
2(i)	$\theta > \left(1 + \frac{pp}{1-\rho}\right)e_1$	$E_{A2(i)} \geq \left[1 + 2\frac{\{1+(p-1)\rho\}\{\theta - e_1e_2\}}{(v-1)\{\theta(\theta-2)+e_1e_2\}}\right]^{-1}$	$\theta > \left(1 + \frac{p\rho^g}{1-\rho^g}\right)e_1$	$E_{B2(i)} \geq \left[1 + 2\frac{\{1+(p-1)\rho^g\}\{\theta - e_1e_2\}}{(v-1)\{\theta(\theta-2)+e_1e_2\}}\right]^{-1}$
2(ii)	$\theta > \left(1 + \frac{pp}{1-\rho}\right)e_1'$	$E_{A2(ii)} \geq \left[1 + 2\frac{\{1+(p-1)\rho\}\{\theta - e_1'e_2'\}}{(v-1)\{\theta(\theta-2)+e_1'e_2'\}}\right]^{-1}$	$\theta > \left(1 + \frac{p\rho^g}{1-\rho^g}\right)e_1'$	$E_{B2(ii)} \geq \left[1 + 2\frac{\{1+(p-1)\rho^g\}\{\theta - e_1'e_2'\}}{(v-1)\{\theta(\theta-2)+e_1'e_2'\}}\right]^{-1}$
2(iii)	$e > \left(1 + \frac{pp}{1-\rho}\right)e_1''$	$E_{A2(iii)} \geq \left[1 + 2\frac{\{1+(p-1)\rho\}\{\theta - e_1''e_2''\}}{(v-1)\{\theta(\theta-2)+e_1''e_2''\}}\right]^{-1}$	$\theta > \left(1 + \frac{p\rho^g}{1-\rho^g}\right)e_1''$	$E_{B2(iii)} \geq \left[1 + 2\frac{\{1+(p-1)\rho^g\}\{\theta - e_1''e_2''\}}{(v-1)\{\theta(\theta-2)+e_1''e_2''\}}\right]^{-1}$

Note: *α is the number of treatments common among the two blocks*

Note: For calculating eigenvalues of $\mathbf{V}^{*}\mathbf{C}_0^{*-}\mathbf{V}^{*\prime}$, without loss of generality we assumed that two observations belong to first two treatments in the first block or to first two treatments in the first two blocks and so on.

Note that, the maximum eigenvalues of $\mathbf{V}^{*}\mathbf{C}_0^{*-}\mathbf{V}^{*\prime}$ is 2 for any value of α. We therefore arrive at the following results.

Theorem 4.3: BIB designs for multi-response experiments with correlation structure A is robust as per connectedness criterion against the loss of all observations from any two plots if and only if

$$\theta > 2\left(1+\frac{p\rho}{1-\rho}\right).$$

Theorem 4.4: BIB designs for multi-response experiments with correlation structure B is robust as per connectedness criterion against the loss of all observations from any two plots if and only if

$$\theta > 2\left(1+\frac{p\rho^{g}}{1-\rho^{g}}\right).$$

Remark 4.3: From the expression of Theorem 4.3 and Theorem 4.4, it is evident that for a correlation coefficient 0 a BIB design will be robust as per connectedness criterion, if and only if $\theta > 2$. Thus, all BIB designs that are robust against the loss of two observations for uni-response experiment will be robust for multi-response experiments when response variables are uncorrelated. For higher values of ρ robustness again depends on the values of p. We also worked out this condition for various values of p and ρ for the designs mentioned earlier. For example for a BIB design with parameters v = 8, b = 8, r = 7, k = 7, and λ = 6, the design is robust for $p \leq 5$ at a correlation co-efficient $\rho = 0.4$.

Lemma 4.1:

$$E_{A1} > E_{A2(i)} > E_{A2(ii)} > E_{A2(ii)} > E_{A2(iii)}$$

and

$$E_{B1} > E_{B2(i)} > E_{B2(ii)} > E_{B2(ii)} > E_{B2(iii)}$$

Remark 4.4: from Lemma 4.1, we see that under both the correlation structures, efficiency factor is maximum for case 1. Thus if the design is robust as per efficiency criterion for the case 1, the design remains robust for the loss of all observations from any two plots. Again for correlation co-efficient 0, the case will become for the case of uni-response experiment. For higher value of ρ, we worked out the efficiency under both the correlation structures for the case 1 and for various values of p and ρ. For example, for a BIB design with parameters v = 11, b = 11, r = 6, k = 6 and λ = 3, under correlation structure A, the efficiency of the design is 0.87 for $p \leq 5$ at $\rho = 0.4$.

5 Discussion

In the present paper the robustness of multi-response experiments conducted in a BIB design against missing data has been investigated. Since in multi-response experiments, the response variables are correlated among themselves, various correlation structures have been considered while studying robustness property. However, the study is confined to some particular cases only. Two cases are considered. In the first case all the observation belonging to a particular plot are missing, while in the second case, all observations belonging to any two plots are missing have been considered. Connectedness property has been studied for various values of p and ρ, *i.e.*, the number of response variables and the value of correlation coefficient. It has been observed that the design becomes disconnected with the increase values of p and ρ. To know whether a design is robust as per efficiency criterion, we have to know the values of p and ρ. These efficiencies values are calculated under different correlation structures also. When g = 1, it becomes, the correlation structure A. The efficiencies are calculated for those designs that are robust as per connectedness criterion. We defined the robustness as per efficiency criterion for a particular design when this efficiency value is greater than or equal to 0.80. For any lower values of p and ρ or for any combination of these maximum values of p and ρ, these efficiency values are greater than or equal to the values calculated.

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