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SYMBOLIC DIRECT PRODUCT DESIGNS

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ABSTRACT

Symbolic direct product of matrices has been applied to the incidence matrices of block designs to define the symbolic direct product designs. The properties of the derived design have been studied to see how the properties of the component designs are carried into the new design.

Keywords: Symbolic direct product (SDP), Binary design, Inter- and Intra-group balanced block design, Ternary design, Optimum design.

AMS Subject Classification : 62 x10

1. Introduction :

The method of symbolic direct product of matrices was first used by Chakravorti (1956) to generate orthogonal asymmetrical fractional factorial plans from known solution of orthogonal plan of symmetrical factorial designs. Raktoe, Hedayat and Federer (1981 page 187) termed it direct product. Here we have used the nomenclature symbolic direct product (SDP) to distinguish it from the ordinary direct product of matrices. New designs have been formed by using the operation of SDP to the incidence matrices of two designs. In Section 2 we have given the analysis of the derived design and have shown how the C-matrix of the derived design is related to those of the component designs. We have exploited these relationships to study the optimality of the derived designs in section 4. In section 3, some combinatorial aspects of these derived designs have been studied.

Let us give the definitions.

Definition – 1.1

Let A = (a_1, a_2, ..., a_n) and B = (b_1, b_2, . . ., b_n) be respectively m $\times$ n and p $\times$ q matrices. Then the symbolic direct product $A \otimes B$ of order (m + p) $\times$ nq of the matrices A and B is defined by

$$A \otimes B = \begin{Bmatrix} a_1 & a_1....a_1 \; a_2....a_2 \; ....a_n....a_n \\ b_1 & b_2....b_2 \; b_1....b_2 \; ....b_n....b_n \end{Bmatrix}$$

Definition 1.2 :

Let $N_1^{v_1 \times b_1}$ = (n_1, n_2, ..., n_{b_1}) and $N_2^{v_2 \times b_2}$ = (m_1, m_2, ..., m_{b_2}) be the incidence matrices of two designs d_1 and d_2 respectively. Then the design d whose incidence matrix N is given by $N_1 \otimes N_2$ is called the Symbolic Direct Product Design (SDPD) of d_1 and d_2.

It is clear from the definition that the symbolic direct product design d of the designs d_1 and d_2 is obtained by taking the treatments of every block of d_2 into every block of d_1. Again if the b_1 columns of N_1 are assumed to give a fractional design of a v_i-factor experiment i = 1, 2, in a single block then d is Kronecker product design (Vartak, 1955) of the fractional designs. It may be seen that if the vectors in $A \otimes B$ be scalars, then we get the SDP in the sense of Kurkjian and Zelen (1962).

2. Analysis of the Symbolic Direct Product Design :

Let d be a binary design with the parameters v_i, b_i, r_i, k_i and $\lambda_{jj}^{(i)}$, i = 1, 2. Then the design d = $d_1 \otimes d_2$ has the following parameters.

$v = v_1 + v_2$, $b = b_1 b_2$, $k = k_1 + k_2$

r_{10} = replication of any treatment t_{1j_1} of $d_1 = b_2 r_1$

r_{20} = replication of any treatment t_{2j_2} of $d_2 = b_1 r_2$

$\lambda_{jj'}^{(1)}$ = replication of any treatment t_{1j} and $t_{1j'}$ of $d_1 = b_2 \lambda_{jj'}^{(1)}$

$\lambda_{jj'}^{(2)}$ = replication of any treatment t_{2j} and $t_{2j'}$ of $d_1 = b_2 \lambda_{jj'}^{(2)}$

$\lambda_{jj'}$ = replication of any treatment t_{1j} of d_1 and $t_{2j'}$ of $d_2 = r_1 r_2$ (2.1)

Let $C_{11}^{v_1 \times v_1} = b_2 r_1 I_1 - K^{-1} b_2 N_1 N_1'$,

$$C_{22}^{v_2 \times v_2} = b_1 r_2 I_2 - K^{-1} b_1 N_2 N_2'$$

and $C_{12}^{v_1 \times v_2} = C_{21}' = -K^{-1}(r_1 r_2) E_{12}$

where $I_i = v_i \times v_i$ identity matrix i = 1, 2 and $E_{12} = v_1 \times v_2$ matrix with all elements unity. Then it can be shown that the C-matrix of the design d is given by

$$C^{v \times v} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \tag{2.2}$$

Let $t' = (t_1', t_2')$ = vector of treatments of the design d

where $t_1' = (t_{11}, t_{12}, ..., t_{1v1})$ = vector of treatments of the design d_1

and $t_2' = (t_{21}, t_{22}, ..., t_{2v_2})$ = vector of treatments of the design d_2

We also define $Q' = (Q_1', Q_2')$, the vector of adjusted treatment totals for the treatments in d, where $Q_1' = (Q_{i1}, Q_{i2}, ..., Q_{iv_i})$, i = 1, 2 and

$$Q_{ij_1} = T_{ij_i} - k_i^{-1} \sum_{\alpha=1}^{b_1} \sum_{a'=1}^{b_2} W_{j_1(\alpha\alpha')} B_{\alpha\alpha'},$$

where $B\alpha\alpha$, represents the (α, α') th block total $W_{j1}(\alpha\alpha')$, represents the element in the j_1th row and (α, α') th column of the incidence matrix N of d, and T_{ij_i} is the treatment total of the treatment T_{ij_i} from design d, $1 \le j_i \le v_i, 1 \le \alpha \le b_1, 1 \le \alpha' \le b_2, 1 \le i \le 2.$

Also let C_1 be the C-matrix of d_1, i = 1, 2. Then the reduced normal equations for $t' = (t_1', t_2')$ become

$$k^{-1}b_2(r_1k_2I_1+k_1C_1)t_1-k^{-1}(r_1r_2)E_{12}t_2=Q_1 \tag{2.3}$$
$$-k^{-1}(r_1r_2)E_{21}t_1+k^{-1}b_1(r_2k_1I_2+k_2C_2)t_2= Q_2$$

Adding separately the first and second set of equations in (2.3) and introducing the non-estimable restriction

$$\hat{t}_0 = \sum_{i-1}^{v} \hat{t}_i = \sum_{j1} \hat{t}_{1j} + \sum_{j2} \hat{t}_{2j2} = \hat{t}_{10} + \hat{t}_{20} = 0 \text{ we get,}$$

$$\hat{t}_{10} = (vr_1r_2)^{-1}Q_{10}, \ \hat{t}_{20} = (r_1r_2v)^{-1}Q_{20} \tag{2.4}$$

where $Q_{i0} = \sum_{j1} Q_{ij}$, i = 1, 2. Since for i = 1, 2, $b_ik_i = v_ir_i$ and $v = v_1 + v_2$, we get from the first set of equations in (2.3)

$$W_1\hat{t}_1 = k(Q_1 + v^{-1}E_{12}Q_2) \tag{2.5}$$

where $W_1 = b_2k_1(C_1 + k^{-1}r_1k_2I_1)$

Now we assume that both d_1 and d_2 are connected so that the characteristic roots of C_i are $\lambda_{11}=0, \lambda_{12}>0, ..., \lambda iv_i > 0$, and the corresponding orthogonal characteristic vectors are $\eta_{i1}, \eta_{i2}, ..., \eta_{iv_i}$,1 < i < 2. Then it is easy to see that

$$b_2k_1(\lambda_{1j} + r_1k_2k^{-1})$$

(>0) is a characteristic root of W_1 with the characteristic vector , λ_{ij} , 1 < j < v_1 . Therefore W_1 is positive definite so that

$$\hat{t}_1 = kW_1^{-1}\left(Q_1 + v^{-1}E_{12}Q_2\right) \tag{2.6}$$

In the same way

$$\hat{t}_2 = kW_2^{-1}\left(Q_2 + v^{-1}E_{21}Q_1\right) \tag{2.7}$$

where $W_2 = b_1k_2(C_2 + k^{-1}r_2k_1I_2)$ which is also a p. d. matrix. Therefore the adjusted treatment sum of squares (with d. f. v – 1) is given by

$$\hat{t}'Q = \hat{t}_1'Q_1 + \hat{t}_2'Q_2 = k(Q_1'W_1^{-1}Q_1 + Q_2'W_2^{-1}Q_2) \tag{2.8}$$

Hence the ANOVA table for testing $H_0 : t_1 = t_2 = . . . = t_v$ can be easily constructed.

Again as we have assumed C_i, the C-matrix of d_i has the ch. roots $\lambda_{i1} = 0, \lambda_{12} > 0, ..., \lambda_{iv_i} > 0$, with the corresponding ch. vectors $\eta_{i1}, \eta_{i2}, ..., \eta_{iv_i}$, where, $\eta_{i1} = v^{-1/2}.1_i$, then it can be shown that the C-matrix of d has the following ch. roots and orthonormal ch. vectors.

Characteristic root	Characteristic vector
(1) $\mu_1 = 0$	$v^{-1/2}\underset{\sim}{1}$
(2) $\mu_2 = k^{-1} b_2 (r_1 k_2 + k_1 \lambda_{1j})$	$\eta'_j = (\eta', O'), 2 \le j \le v_1 \sim ij \sim 2$
(3) $\mu_{v1} + 1 = k^{-1} r_1 r_2 v$	$\eta'_{v_1+1} = (vv_1)^{-\frac{1}{2}} V_2^{\frac{1}{2}} 1'_1 - (VV_2)^{-\frac{1}{2}} V_1^{\frac{1}{2}} 1'_2$
(4) $\mu_{v_1+j} = k^{-1} b_1 (r_2 k_1 + k_{22j)}$	$\eta'_j = (0'_1 \eta'_{2j}), 2 \le j \le v_2$ (2.9)

where 1_i and 1 are $v_i \times l$ and vxl vectors of unities respectively, i = 1, 2, $O'_1 = 1 \times v_i$ null matrix, i = 1, 2. Thus we get the following Lemma.

Lemma 2.1 : d is connected if d_1 and d_2 are connected and conversely.

3. Some combinatorial properties

Nair and Rao (1942) introduced inter-and intra-group balanced block design (IIGBBD) which was also studied by Corsten (1962), Adhikary (1965) and others. Let there be v treatments divided into m groups $G_1, G_2, \ldots, G_m$ having respectively $v_1, v_2, \ldots\ldots, v_m$ treatments (obviously $v = \sum^{m} v_i$). Then if the treatments are arranged in b blocks each of size k such that any treatment belonging to G_i occurs r_i times and any two treatments of G_i occur together in λ_{ii} blocks, any treatment of G_i occurs with any treatment of G_j in λ_{ij} blocks of the design $1 \le i \ne j \le m$. We now generalize this IIGBBD to inter- and intra-group partially balanced block design (IIGPBBD). Let the v_i treatments in G_i follow a partially balanced association scheme S_i, $1 \le i \le m$ and the $v_i v_{i'}$, pairs of treatments from G_i and $G_{i'}$ can be grouped into , m_{ii}' sets $1 \le i \ne i \le$ m. Then we have the following definition.

Definition – 3.1 : A set of v treatments arranged in b blocks of size k is said to be an IIGPBBD if it satisfies the following conditions :

i) Every treatment belonging to G_i occurs in r_i blocks, $1 \le i \le m$.

ii) If any two treatments of G_i are jth associate, then they occur together in $\lambda_j^{(i)}$ blocks, j = 1, 2, ..., m_1, $1 \le i \le m$.

iii) Any treatment t belonging to G_i and any treatment belonging to $G_{i'}$, occur together in $\lambda_{ii'}^{(\alpha)}$, blocks, if the two treatments are chosen from the αth set of $v_i v_{i'}$, pairs of treatments, $1 \le \alpha \le m_{ii'}$, $1 \le i \ne i' \le m$.

Let us denote such a design by IIGPBBD $\left(\sum v_i, b, k, r_1 r_2, \ldots\ldots, r_m, \lambda_j^{(i)}, \lambda_{ii}^{(\alpha)}\right)$. Obviously if $\lambda_j^{(i)}$ and $\lambda_{ii}^{(\alpha)}$, be independent of j and α respectively then the IIGPBED reduces to IIGBED. We now state the following theorem which is easy to prove.

Theorem 3.1 : If we have two PBIBD's d_1 and d_2 with parameters

$d_i (v_i, b_i, r_{i0}, k_i, \lambda_{ij}, j = 1, 2, ..., m_i), 1 \le i \le 2$. Then the SDPD of d_1 and d_2 is an IIGPBBD with parameters

$v = v_1 + v_2$, $b = b_1 b_2$, $r_1 = r_{10} b_2$, $r_2 = r_{20} b_1$, $k = k_1 + k_2$,

$$\gamma_j^{(i)} = \lambda_{1j} . b_j^{(2)}, \lambda_j^{(2)} = \lambda_{2j} . b_1, 1 \le j \le m_i, \lambda_{12} = r_{10} . r_{20}$$

Corollary 3.1 : If d_i be a BIBD with parameters d_i $(v_i, b_i, r_{i0}, k_i, \lambda_i)$, i = 1, 2, then $d = d_1 \otimes d_2$ is an IIGBBD with parameters $v = v_1 + v_2$

$$b = b_1 b_2, r_1 = r_{20} . b_1, \lambda_{11} = \lambda_1 b_2, \lambda_{22} = \lambda_2 b_1$$

and $\lambda_{22} = \lambda_2 b_1$ and $\lambda_{12} = r_{10} . r_{20}$..

Corollary 3.2 : If d_1 and d_2 be two BIBD's with same parameters $(v_0, b_0, r_0, k_0, \gamma_0)$ but with different symbols then $d = d_1 \otimes d_2$ is a semiregular group divisible design (SRGDD).

Proof : Let the treatment of d_1 form a group G_1, i =

1, 2 and let any two treatments of the same group are first associate and any two treatments of different groups are second associate. Then it is easy to see that $d = d_1 \otimes d_2$ is a GDD with two groups containing v treatments each, with parameters $v=2v_0, b=b_0^2, k=2k_0, r=b_0r_0, \lambda_1=b_0\lambda_0$ and $\lambda_2 = r_0^2$. It is clearly seen that $rk-v\lambda_2 = 0$, so that the design is a SRGD.

Corollary – 3.3 : If d_1 and d_2 be two IIGBBD's then $d = d_1 \otimes d_2$ is again a IIGBBD.

Theorem 3.2 : If d_1 and d_2 represent two m-associate PBIBD's which are same except for the treatment symbols then $d = d_1 \otimes d_2$ gives a PBIBD with (m + 1) associate classes.

Proof : Let (i1), (i2), ..., (iv_0) represent the v_0 symbols of the design d_1, which are such that the v_0 treatments of d_1 have an association scheme S_i among themselves, i = 1, 2. Let us define any treatment of d_1 to be an (m + 1)th associate of any treatment of d_2. Also if B_{j0} be the jth association matrix of the treatments of d_1 (and hence of d_2 also) j = 1, 2, ..., m then d is obviously a (m+1) associate PBIBD with the association matrices as

$$B_j = \begin{pmatrix} B_{j0} & 0 \\ 0 & B_{j0} \end{pmatrix} \text{ if } \quad j = 1, 2, ...m, = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \text{ if}$$

j = m + 1

where 0 is a $v_0 \times v_0$ null matrix and E is a $v_0 \times v_0$ matrix with all elements unity.

Partially balanced ternary design was introduced by Paik and Federer (1973). A design d having the incidence matrix $N^{v \times b}$ with elements 0, 1, 2 each occurring at least one in each row, is said to be partially balanced ternary design if

$$NN' = a_0B_0 + a_1B_1 + \ldots + a_mB_m \qquad (3.1)$$

where B_1's ($i \geq 1$) are the association matrices and B_0 is $v \times v$ identity and a_i's are scalars. Now suppose that design d with the incidence matrix N denotes a PBIBD with parameters $v_0, b_0, r_0, \lambda_{10}$ and λ_{20} Then we have the following theorem.

Theorem 3.3 : $d_0 = d \otimes d$ gives a partially balanced ternary design.

Proof : The proof follows by noting that $N_0 = N_1 \otimes N_2 = N_1 + N_2$, $N_1 = N_2 = N$, where + denotes the direct sum of the columns of N. Then it can be shown that $N_0N'_0$ has the same structure as in (3.1).

Corollary 3.3 : If d be BIBD, then $d_0 = d \times d$ is a balanced ternary design) (Tocher 1952).

Remark – 3.1 : This process can be continued to have balanced of partially balanced n-any designs.

4. Optimum properties :

In this section, we have studied the optimum properties of the derived design from those of the component designs.

To start with, let ***D*** (v, b, k) be the collection of all binary equireplicate and proper designs with given parameters v (number of varieties), b (number of blocks) and k (block size). With the usual additive model, let C_d be the coefficient matrix of the design d. Then the theory of optimal experimental design is concerned with the problem of selecting a design which minimises some functional ψ of C_d over all possible designs (for details see Kiefer (1959, 1975)). It is known (Kiefer (1975)) that if for a design d, C-matrix is completely symmetric (c. s.), it is universally optimal. And such designs with C-matrix c. s. exist only for some restricted class of parameter of v, b and k. Cheng (1978) looked into the case where there was no BBD (for which c-matrix is c.s.) and introduced the type 1 and type 2 criteria.

Let $\boldsymbol{B}_{v,0}$ be the class of v×v symmetric matrices with row and column sums all zeroes. Let C_d be a matrix belonging to v_0 where $d \in \boldsymbol{D}$, a class of binary designs with parameters v, b, k and let the ch-roots of C_d be $\lambda_{d1} = 0, \lambda_{d2} > 0, ..., \lambda_{dv} > 0$. Then a design d* will be called type-1 or type-2 optimal if the roots of the associated matrix C_d minimise the functional ψf defined by $\sum_{i=2}^{v} f(\lambda_{di})$, where f is continuous, strictly convex, strictly decreasing having continuous derivatives and being strictly concave for type-I and strictly convex for type-2, *i.e.* $f'<0, f''>0, f'''<0$ for type-I and $f'''' > 0$ for type-2.

Theorem 4.1 : Let $d_i \in \mathbf{D}_i$, *i=1* 2 be type-I or type-2 optimal. Then the SDP design $d = d_1 \otimes d_2$ is also type-I or type-2 optimal in the class of designs

$\boldsymbol{D} = \boldsymbol{D}_1 \otimes \boldsymbol{D}_2 =$

$\{d_0 / d_0 = d_1 \otimes d_2, d_1 \in \mathbf{D}_1, i=1,2\}$

Proof : We note from (2.9) that if x be a root of d_1 and y be a root of d_2 $(x > 0, y > 0)$ then the (+)ve roots of d_0 are given by $g(x) = a + bx$, $w(y) = c + dy$ and another root μ which solely depends on the parameters of d_1 and d_2, where (from Section-2) $a > 0$, $b > 0$, $d > 0$, $c > 0$. Now d_0 would be type i, i = 1, 2 optimal if $\sum_X f(g(x)) + \sum_y f(W(Y)) + f(\mu)$ is minimum for all designs $d_0 \in \boldsymbol{D}$. It is sufficient to prove the first part. Now we can write $\sum_x f(g(x) = \sum_x F(x)$, where F = f(g), g is continuous, monotonically increasing. Therefore it follows that F satisfies all the properties of f( ). But since d_1 is type i opt., $i=1,2. \sum_x F(x)$ is min, which implies that $\sum_x f(g)(x))$ is min. Hence the Theorem is proved.

From the above theorem, the following corollary is immediate.

Corollary 4.1 : The design $d_0 = d_{10} \otimes d_{20}$ where d_{10} and d_{20} are (BIBD's) with respective parameters, is optimum with respect to optimality criteria of type i, i = 1, 2 among all designs $\boldsymbol{D} = \{ d = d_1 \otimes d_2, d_1 \in \boldsymbol{D}_1, d_2 \in \boldsymbol{D}_2$ where $\boldsymbol{D}_i$ is the class of all connected proper (incomplete) block designs, i = 1, 2.}

Corollary 4.2 : IIGBBD with parameters $v = v_1 + v_2$, $b = b_1 b_2$, $k = k_1 + k_2$, $r_1 = r_{10} b_2$, $r_2 = r20.b_1$, $\lambda_{11} = \lambda_1 b_2$, $\lambda_{22} = \lambda_2 . b_1$ and $\lambda_2 = r_{10} r_{20}$ is optimum within SDP designs similar to those in Corollary 4.2 with respect to optimality of type i, i = 1, 2.

Proof : Proof follows from Corollaries 3.1 and 4.1.

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