Visualizing the analysis of variance

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ABSTRACT

A vertical line, which renders equal the areas of two regions bounded by itself and the empirical cumulative distribution function of either the data or their appropriate transformations, depicts the mean, the mean deviation, the mean square deviation or the standard deviation. Here, we extend the vertical line method to visualize all statistics involved in the analysis of variance method.

***Keywords*:** Mean proportional; right triangle; third proportional; weighted average

1. INTRODUCTION

To compare the means of a continuous variable X across $k > 2$ subgroups, the most prevalent technique is the one-way analysis of variance (ANOVA). The method works best under the assumption that the subgroup data are drawn randomly and independently from the respective subpopulations, which are normally distributed with the same variance σ^2. The method, first introduced by Fisher (1925), is now well-known. For detailed exposition, see Dudewicz and Mishra (1988), for example. The summary below introduces the notation and initializations.

Essentially, in the ANOVA method we decompose the (corrected) sum of squares total (SST) in the entire data (all subgroups combined) into two statistically independent (due to Cochran's theorem) components: (1) the sum of squares between (SSB) subgroups, and (2) the aggregated sum of squares within (SSW) all subgroups; that is, SST = SSB + SSW. Then we calculate the mean square between, MSB = SSB / (k-1); the mean square within, MSW = SSW / (n-k); and finally the F-statistic, F = MSB / MSW. The MSW has expectation σ^2, while the MSB has expectation σ^2 plus a fraction 1/(k-1) of the sum of squares of deviations of subgroup means from the overall mean. Therefore, the null hypothesis of equal subgroup means is rejected if the computed F-statistic is too large.

Although the ANOVA method is familiar and commonplace, it is surprising that the literature does not offer a visual representation of it. We hope to fill this gap. To present the main results, we briefly review the visualization of the mean, the mean deviation (MD), the mean square deviation (MSD) and the standard deviation (SD) of a single sample in Section 2. Details are found in Sarkar and Rashid (2016 d). In Section 3, we apply the visualization technique on each subgroup data and also on the collection of all k subgroup means to visualize a (scaled) SSB. In Section 4, we visualize a (scaled) SSW and a (scaled) SST; and hence, the F-statistic. Section 5, extends the visualization method to a two-way ANOVA without interaction. Section 6 gives a few concluding remarks and directions of future research. All figures in this paper are drawn using the statistical software R.

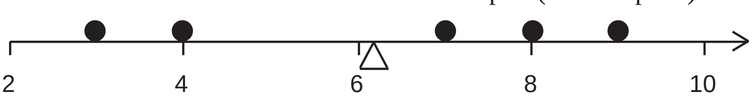
2. VERTICAL LINES DEPICTING MEAN, MD, MSD AND SD

The mean is the most common measure of center. See Pollatsek *et al*. (1981) and Lesser *et al*. (2014). The mean of a set of n numbers $\{x_1, x_2, ..., x_n\}$ is defined by $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ (1)

The mean is interpreted as the location of a fulcrum that balances the dot plot. See Watier, *et al*. (2011). Figure 1 depicts the dot plot of the data in Example 1 and its mean.

Example 1. The number of trips an ambulance made on five days are: 7, 4, 8, 3, 9.

Figure 1. The mean is shown as a fulcrum that balances the dot plot (in Example 1)



An alternative interpretation of the mean involves the empirical cumulative distribution function (ECDF) of the data, which is a step function given by $F(x) = N(x)/n$, where $N(x)$ is a count of data values that are *no more than x*. see figure 2.

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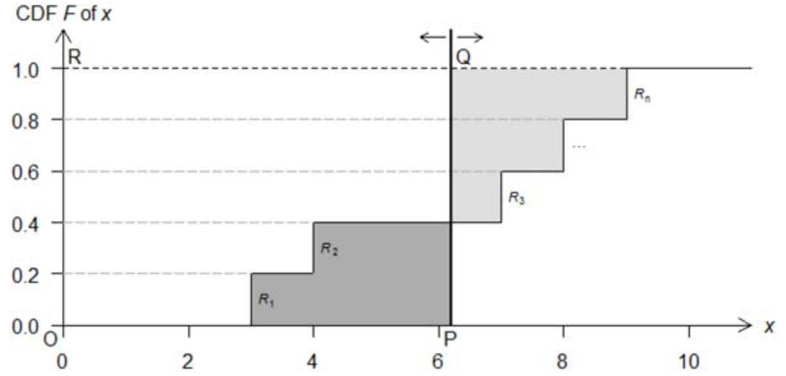
Figure 2. The mean is shown as a vertical line PQ that renders equal the two shaded areas



Consider the inverse-ECDF (or rather the inverse mapping) $x = F^{-1}(y)$, $y \in [0,1]$. Note that

$$\int_0^1 F^{-1}(y)\,dy = \int_0^{1/n} F^{-1}(y)\,dy + \ldots + \int_{1-1/n}^{1} F^{-1}(y)\,dy = \frac{x_1}{n} + \ldots + \frac{x_n}{n} = \bar{x} = \int_0^1 \bar{x}\,dy$$

In other words, the sum of the areas of rectangles $R_1, R_2, \ldots R_n$, each of height $1/n$, equals the area of one single rectangle OPQR (of height 1) bounded by y=0, x=0, y=1 and $x=\bar{x}$.

Therefore, the mean can be visualized as a vertical line PQ that equalizes the areas of two regions (shown in figure 2 by two types of shadings) bounded by the vertical line itself, the two horizontal lines $y = 0$ and $y = 1$, and the ECDF F. This vertical line PQ, representing the mean, can be found by using an Euclidean construction given in Sarkar and Rashid (2016 d).

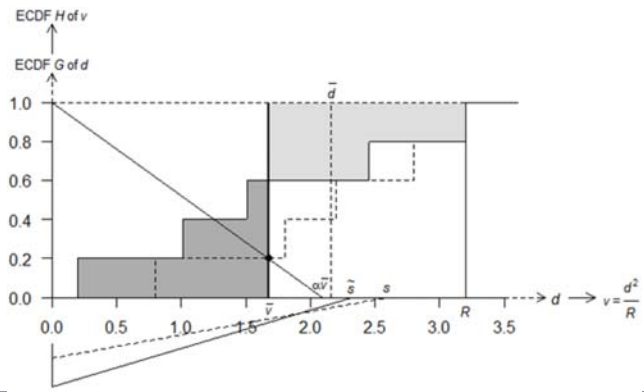
In figure 3 we show the MD and the MSD of a set of numbers as vertical lines. We also show the root MSD (RMSD) and SD as the mean proportional between two segments.

The deviations of the n given numbers from their mean are $d_i = |x_i - \bar{x}|$. The average of *all deviations* from the mean is called the MD, and is given by

$$MD = \frac{1}{n}\sum_{i=1}^{n}|x_i - \bar{x}| = \frac{1}{n}\sum_{i=1}^{n} d_i \qquad (2)$$

To visualize the MD, which is an *average*, we first construct the ECDF G of all these deviations, by reflecting the portion of F to the left of the vertical line $x=\bar{x}$ at the mean about this line, with the reflection falling to the right side of the line. The resulting rectangles of height $1/n$ to the right of the mean, when sorted from the smallest (in width) at the bottom to the largest at the top, yield G. See the dashed steps in figure 3. To find the MD, we search for another vertical line (the dashed vertical line $d = \bar{d}$ in figure 3) that equalizes the areas to its left and to its right and bounded by G and horizontal lines $y = 0$ and $y = 1$.

Figure 3. The dotted ECDF G of deviations yields the MD, and the solid ECDF H of scaled squared deviations yields the RMSD $\tilde{s} = \sqrt{\bar{v}.R}$ and the SD $s = \sqrt{\alpha\,\bar{v}.R}$, where $\alpha = n/(n-1)$



Let us next review the geometric visualization of the MSD and the SD, all shown in figure 3. The sample variance and the sample MSD of the set of numbers are defined by

$$\text{Variance} = s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \tag{3}$$

and

$$MSD = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \tag{4}$$

The sample variance is just a multiple $\alpha = n/(n-1)$ of the sample MSD, with interrelation given by

$$s^2 = \frac{n}{n-1}\tilde{s}^2 = \alpha\,\tilde{s}^2 \tag{5}$$

Taking the positive square roots of (3) and (4), we obtain the sample SD S and the sample RMSD $\tilde{s}$ respectively. For various interpretations of $\tilde{s}$ and s, see Sarkar and Rashid (2016 a–c). For a Euclidean geometric visualization of $\tilde{s}$ and s, detailed in Sarkar and Rashid (2016 d), we construct the ECDF H of (scaled) squared deviations as explained below.

The ECDF G of the deviations, form a collection $\mathcal{R}$ of rectangles whose widths equal the deviations and heights equal $1/n$. We transform each rectangle in $\mathcal{R}$ by changing only its width, but keeping it left aligned at $d = 0$ and maintaining its height unaltered as follows:

Choose R to be a suitable positive magnitude (for example, let R be the largest deviation from the mean), and fix it. Let d be the width of any one rectangle in $\mathcal{R}$. We construct the third proportional to R and d; that is, we find v such that $R : d = d : v$. Thus, a rectangle of width d changes into a new rectangle of width $v = d^2 / R$. When we apply this width-transformation to each rectangle in $\mathcal{R}$, using the same R, we obtain the ECDF H of the scaled (that is, divided by R) squared deviations. Henceforth, the horizontal axis also represents $v = d^2/R$.

Over H in figure 3, we superimpose the vertical line $v=\bar{v}$ that equalizes the areas of the shaded regions to its two sides and bounded by itself, two horizontal lines $y = 0$, $y = 1$ and H. Then the vertical line $v=\bar{v}$ represents the scaled MSD, $\tilde{s}^2 / R$. Finally, to obtain the (unscaled) RMSD, we construct the mean proportional between $\bar{v}$ and R, as explained in the next paragraph, since

$$\sqrt{\bar{v}\,R} = \sqrt{\left(\frac{1}{n}\sum_{i=1}^{n}\frac{d_i^2}{R}\right)R} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}d_i^2} = RMSD \tag{6}$$

Indeed, to construct the mean proportional $\sqrt{ab}$ between a and b (with $a > b > 0$), we draw a right triangle with hypotenuse $(a + b) / 2$ and one leg $(a - b) / 2$. Then the other leg of that right triangle has length $\sqrt{ab}$. Such a right triangle, showing the mean proportional between $\bar{v}$ and R, is depicted (with a solid hypotenuse) below the horizontal axis in Figure 3.

Also, in figure 3, if we join R = (0, 1) to ($\bar{v}$, $1/n$) by a line and extend it to meet the horizontal axis, we obtain a scaled variance $\frac{n\bar{v}}{n-1} = \alpha\bar{v} = \frac{s^2}{R}$. The mean proportional between $\alpha\bar{v}$ and R gives the (unscaled) SD s. See the other triangle (with a dotted hypotenuse) below the horizontal axis in figure 3.

Expression (6) guarantees that we can choose R to be any arbitrary positive number since its effect is eventually eliminated, and we obtain the unscaled RMSD $\tilde{s}$ and the unscaled SD s.

However, to avoid needing additional space to draw H and to ensure precision in drawing, we recommend choosing R to be the largest deviation from the mean. Alternatively, if one chooses R to be the MD, the above described geometric visualization also vividly demonstrates that $s \geq \tilde{s} \geq MD$.

3. SUM OF SQUARES BETWEEN

In the one-way ANOVA set up, from k subpopulations we have drawn k independent samples of sizes n_1, n_2, ..., n_k respectively. Let $n = n_1 + n_2 + ... + n_k$ be the total sample size. Let us denote the X values in subgroup i ($1 \leq i \leq k$) by $\{x_{i,j}; j = 1, 2, ..., n_i\}$. Consider example 2.

Example 2. Students from three high schools participated in a math contest, and their scores (ordered from the lowest to the highest) are as follows:

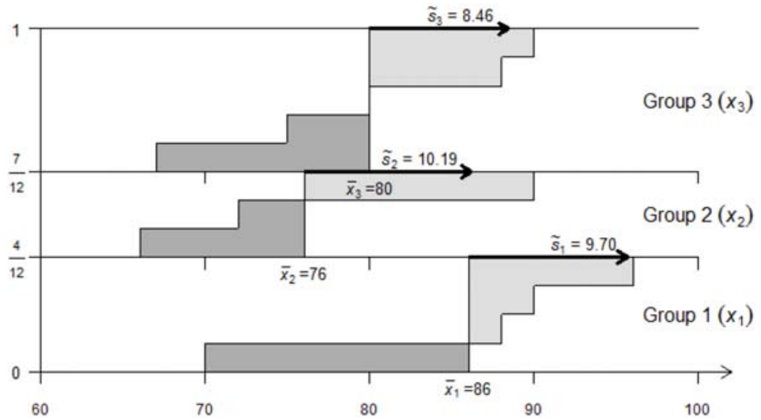
Group 1	School A	70, 88, 90, 96
Group 2	School B	66, 72, 90
Group 3	School C	67, 75, 80, 88, 90

We are interested in finding out if there is a significant difference among the schools in terms of the mean scores obtained by its participants. We summarize the information within each school by reporting their sample sizes, means and RMSD's:

$$n_1 = 4, \bar{x}_1 = 86, \tilde{s}_1 = \sqrt{94};\ n_2 = 3, \bar{x}_2 = 76, \tilde{s}_2 = \sqrt{104};\ n_3 = 5, \bar{x}_3 = 80, \tilde{s}_3 = \sqrt{71.6}$$

We begin by showing the summary statistics in figure 4, using the methods of section 2. Note that we have stacked the CDF's of the subgroups and rescaled their heights (proportionally to the sample sizes) so that their total height is one. Also, we have depicted each RMSD as the length of an arrow drawn at the top of each CDF proceeding to the right of the respective mean vertical line.

Figure 4. Scaled CDF's, means and RMSD's of scores from three schools (in example 2)



Any standard statistical software package will produce a one-way ANOVA table. For example, using R, we obtain:

Table 1. One-way ANOVA for example 2

Sources	Df	SS	MS	F	p-value
School (Between)	2	180	90.0	0.774	0.489
Within	9	1046	116.2	-	

We want to visualize all statistics involved in the one-way ANOVA including scaled versions of SSB and SSW, and the (unscaled) F-statistic. The last two items require the subgroup RMSD's, and will be dealt with in section 4. In this section, we utilize the subgroup means and describe in the next three paragraphs a geometric visualization of (a scaled) SSB defined by

$$\mathrm{SSB} = \sum_{i=1}^{k} n_i (\bar{x}_i - \bar{\bar{x}})^2 \tag{7}$$

First, we draw (see figure 5a) the 'ECDF' $F_{\bar{x}}$ of $\bar{x}_i$. It is a 'step function' with step height $\frac{n_i}{n}$ at the group mean $\bar{x}_i$. Indeed, Figure 5a is obtained from figure 4 by keeping only the vertical line segments representing the subgroup means. Note that the steps in figure 5a are not sorted from the shortest at the bottom to the widest at the top, nor are they of equal heights. Still, the methods of Section 2 continue to work. To visualize the overall mean, we simply draw a vertical line on to this 'ECDF' $F_{\bar{x}}$ that equalizes the areas to its left and right and bounded by itself, $y = 0$, $y = 1$ and $F_{\bar{x}}$. Of course, this vertical line must be drawn at the weighted average of $\bar{x}_i$ given by $\bar{\bar{x}} = \sum_{i=1}^{k} \frac{n_i}{n} \cdot \bar{x}_i$ (8)

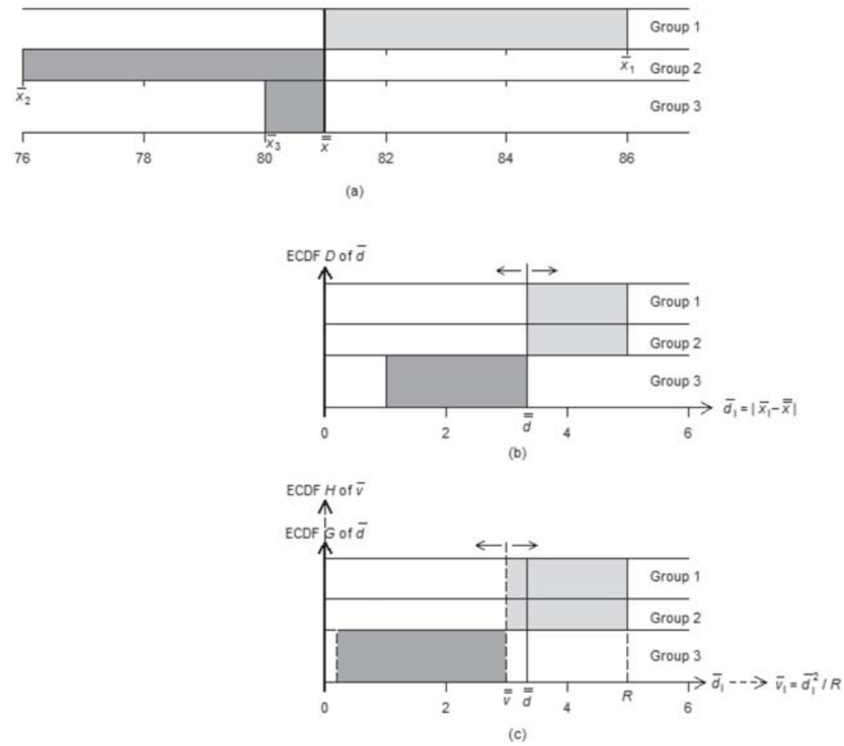
Next, we construct (see figure 5b) the ECDF $G_{\bar{x}}$ of deviations of $\bar{x}_i$'s from $\bar{\bar{x}}$ by reflecting the portion of $F_{\bar{x}}$ to the left side of the vertical line $x = \bar{\bar{x}}$ about this line, with the reflection falling to the right side, and thereafter (optionally) sorting the rectangles from the smallest (in width) at the bottom to the longest at the top, and changing the horizontal axis to show $\bar{d}_i = \left|\bar{x}_i - \bar{\bar{x}}\right|$. We obtain the MD of $\bar{x}_i$ from $\bar{\bar{x}}$ by drawing another vertical line on to $G_{\bar{x}}$ to equalize the areas to its left and right. Again, this vertical line must be drawn at the weighted average of $\bar{d}_i$'s given by

$$\bar{\bar{d}} = \sum_{i=1}^{k} \frac{n_i}{n} \cdot \bar{d}_i = \sum_{i=1}^{k} \frac{n_i}{n} \cdot |\bar{x}_i - \bar{\bar{x}}| \tag{9}$$

Finally, (see figure 5c) choose a suitable scale R (our choice is the largest deviation $\bar{d}_{\max}$; alternatively, one can choose the MD $\bar{\bar{d}}$ and obtain the third proportional to each deviation, using the method illustrated in Figure 3. Thus we obtain the 'ECDF' $H_{\bar{x}}$ of the scaled squared deviations $\bar{v}$'s. Superimposing a vertical line that equalizes the areas to its left and right and bounded by $y = 0$, $y = 1$ and $H_{\bar{x}}$, we can visualize SSB/(nR), since

$$\bar{\bar{v}} = \sum_{i=1}^{k} \frac{n_i}{n} \cdot \bar{v}_i = \sum_{i=1}^{k} \frac{n_i}{n} \cdot \frac{d_i^2}{R} = \sum_{i=1}^{k} \frac{n_i}{n} \cdot \frac{(x_i^2 - \bar{\bar{x}})^2}{R} = \frac{\mathrm{SSB}}{nR}$$

Figure 5. The group means in Example 2 yield (a) the overall mean $\bar{\bar{x}}$, (b) the MD $\bar{\bar{d}}$ of the deviations of group means from $\bar{\bar{x}}$, and (c) a scaled SSB $\bar{\bar{v}} = \dfrac{SSSB}{nR}$, where $R = \max d_i$



4. SUM OF SQUARES WITHIN AND F-STATISTIC

In this section we utilize the subgroup RMSD's $\tilde{s}_i$'s to obtain (a scaled) SSW, which is the sum of the total squared deviations within each subgroup, defined by

$$\mathrm{SSW} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{i,j} - \bar{x}_i)^2 = \sum_{i=1}^{k} n_i \tilde{s}_i^2 \tag{10}$$

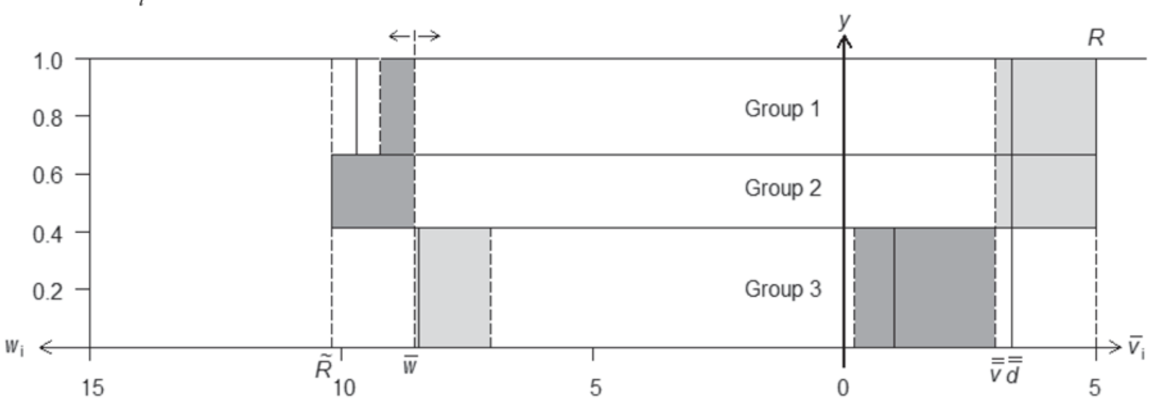
Below we describe a geometric visualization of a scaled SSW and the (unscaled) F-statistic. We also visualize the coefficient of determination, which is the proportion of total variation in *X*-values attributed to the ANOVA model.

In figure 6, we first redraw figure 5(c); and then for each i $(1 \le i \le k)$, we draw a rectangle of height n_i/n and width $\tilde{s}_i$, carefully right-aligning it at $\bar{d} = 0$ and positioning it directly to the left of the rectangle of height n_i/n and width $\bar{d}_i = |\bar{x}_i - \bar{\bar{x}}|$ on the right side of $\bar{d} = 0$. We treat the horizontal axis as *positive* in both the right and the left directions.

Next, choose a suitable scale $\tilde{R}$ (our choice is the largest RMSD or max $\tilde{s}_i$; alternatively, one can choose the same R chosen in the previous section), and obtain the third proportional to each RMSD $\tilde{s}_i$ (the first number always being $\tilde{R}$). Thus, for each i, we obtain a new rectangle (right-aligned at $\bar{d} = 0$ and with a dashed left boundary) of height n_i/n and width $w_i = \tilde{s}_i^2 / \tilde{R}$. Then we find the (dashed) vertical line $w = \bar{w}$ that renders equal the areas to its left and right (as shaded in Figure 6) depicting SSW/$(n\tilde{R})$, since

$$\bar{w} = \sum_{i=1}^{k} \frac{n_i}{n} \cdot w_i = \sum_{i=1}^{k} \frac{n_i}{n} \cdot \frac{\tilde{s}_i^2}{\tilde{R}} = \frac{\text{SSW}}{n\tilde{R}}$$

Figure 6. The group RMSD's in example 2, shown to the left of the *y*-axis, yield a scaled SSW as $\bar{w}$ = SSW/$(n\tilde{R})$, where $\tilde{R} = max\, \tilde{s}_i$



Finally, to visualize the F-statistic, we rewrite it as follows (using $\beta = R / \tilde{R}$) :

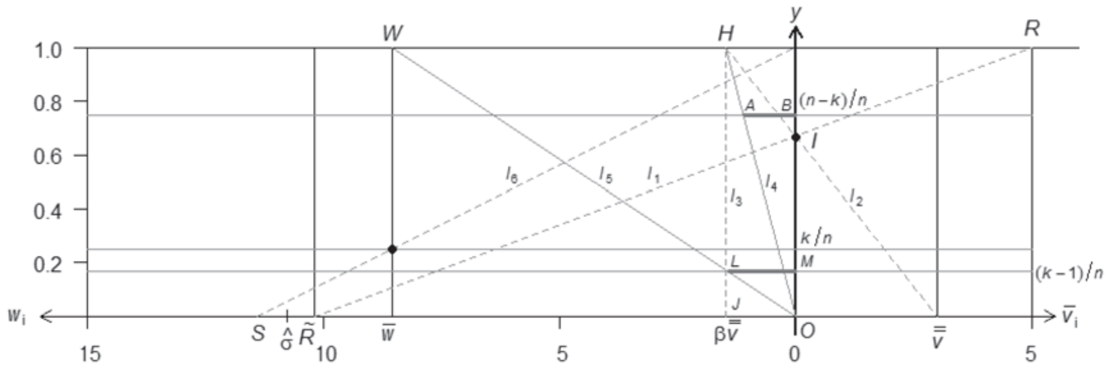
$$F = \frac{\text{MSB}}{\text{MSW}} = \frac{\text{SSB}}{\text{SSW}} \cdot \frac{n-k}{k-1} = \frac{\frac{R}{\tilde{R}} \cdot \frac{\text{SSB}}{nR}}{\frac{\text{SSW}}{n\tilde{R}}} \cdot \frac{n-k}{k-1} = \frac{\beta \bar{\bar{v}} \cdot \frac{n-k}{n}}{\bar{w} \cdot \frac{k-1}{n}} = \frac{N}{D}, \text{ say} \tag{11}$$

We construct (i) the numerator N and (ii) the denominator D in (11) by drawing a sequence of lines (see figure 7):

(i) We draw a line l_1 joining ($\tilde{R}$, 0) and (R, 1), and let it intersect the *y*-axis at I. We draw a line l_2 joining ($\bar{\bar{v}}$, 0) and I, and extend it to meet the horizontal line $y = 1$ at H. From H we drop a perpendicular l_3 meeting the horizontal axis at J. Then segment OJ has length $\beta \bar{\bar{v}} = \frac{R}{\tilde{R}} \bar{\bar{v}} = \frac{\text{SSB}}{n\tilde{R}}$. We draw a horizontal line h_{n-k} through B = (0, $(n-k)/n$), and a line l_4 joining O = (0, 0) to H = $(\beta \bar{\bar{v}}, 1)$ cutting h_{n-k} at A. Segment AB has length N.

(ii) We draw a horizontal line h_{k-1} through M = (0, k–1)/n), and a line l_5 joining O = (0, 0) to W = ($\bar{w}$, 1) cutting h_{k-1} at L. Segment LM has length D. The ratio AB : LM, which is free of R and $\tilde{R}$, is the F-statistic = 0.7744.

Figure 7. Visualizing the F-statistic as a ratio AB : LM (for the data in example 2). Also, one can visualize $\sqrt{\text{MSW}} = \hat{\sigma}$ as the mean proportional between OS and O$\tilde{R}$



Additionally, we can visualize the root MSW (RMSW), which is an estimate of the common population SD σ, as follows : Join (0, 1) to $Z = (\bar{w}, k/n)$ by a line l_6 and extend it to cut the horizontal axes at S. Segment OS has length $\bar{w}\, n/(n-k)$. Taking the mean proportional between OS and O$\tilde{R}$ (as shown in figure 3) one can find $\sqrt{\text{MSW}} = \hat{\sigma}$.

Finally, we can also visualize the (scaled) SST and the (unscaled) coefficient of determination of the ANOVA model. Recall that we have complete freedom to choose R and $\tilde{R}$ as we please. In case we choose $\tilde{R} = R$, we can visualize SST/($n\tilde{R}$) as SSB/($n\tilde{R}$) + SSW/($n\tilde{R}$) = $\bar{\bar{v}} + \bar{w}$. However, even when $\tilde{R} \neq R$, we have shown in figure 7 how to construct segment OJ of length $\beta\bar{\bar{v}} = \frac{R}{\tilde{R}}\bar{\bar{v}} = \frac{\text{SSB}}{n\tilde{R}}$. So, we can visualize SST/($n\tilde{R}$) as $\beta\bar{\bar{v}} + \bar{w}$. The coefficient of determination SSB/SST can be visualized as $\beta\bar{\bar{v}} : \left(\beta\bar{\bar{v}} + \bar{w}\right)$.

Let us close this section by exhibiting in one picture how to obtain the F-statistic starting from the subgroup means and the RMSD's. To avoid repetition, we do so after modifying example 2.

Example 3. Consider a modification of the data in example 2. Suppose that upon reevaluation of contest papers, the scores of each participant in School B decreased by 10, in School C the scores increased by 6, and in School A they remained unchanged. Then the summary statistics are :

$$n_1 = 4, \bar{x}_1 = 86, \tilde{s}_1 = \sqrt{94};\ n_2 = 3, \bar{x}_2 = 66, \tilde{s}_2 = \sqrt{104};\ n_3 = 5, \bar{x}_3 = 86, \tilde{s}_3 = \sqrt{71.6}$$

How did the F-statistic change? Using R, we show the new results in table 2.

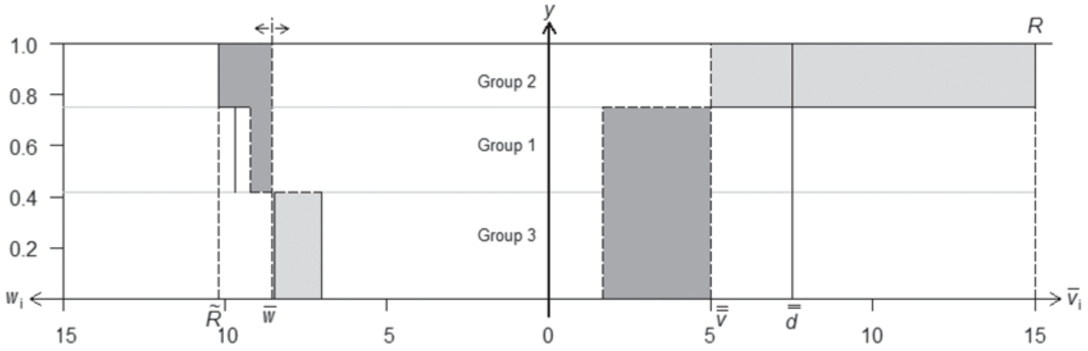
Table 2. One-way ANOVA for example 3

Sources	Df	SS	MS	F	p-value
School (Between)	2	900	450.0	3.872	.0612
Within	9	1046	116.2	-	

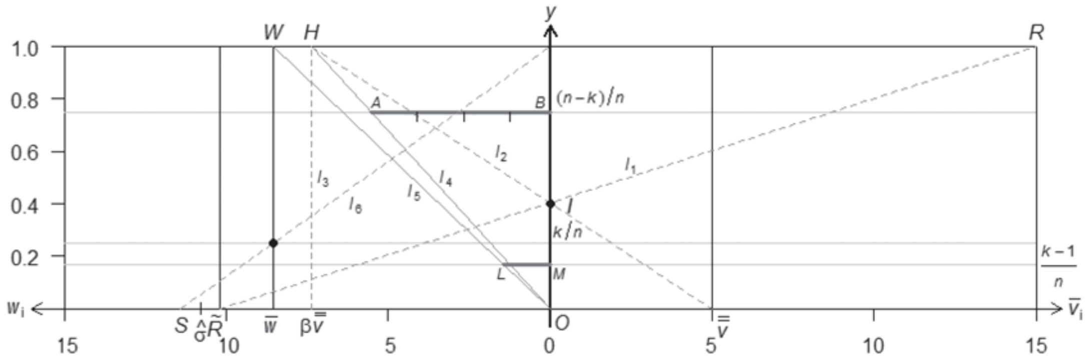
To visualize the F-statistic, note that even though some within group means have changed, the total score of all 12 participants (and hence the overall mean score) has not changed; we still have $\bar{\bar{x}} = 81$. Also, since the spread within each subgroup is unaltered, the SSW remains unchanged. However, the subgroup means have become more spread out from the overall mean. In fact, the SSB has increased fivefold, and so has the F-statistic. See Figure 8.

Figure 8. (a) The group means in example 3 yield the MD $\bar{\bar{d}}$ and the scaled SSB, $\bar{\bar{v}}$ = SSB/(nR) with R = max $\left|\bar{x}_i - \bar{\bar{x}}\right|$, on the right side of the vertical axis. The group RMSD's yield a scaled SSW, $\bar{w}$ = SSW/($n\tilde{R}$) with $\tilde{R}$ = max$\tilde{s}_i$, on the left side of the vertical axis.

(b) The F-statistic is the ratio AB : LM ; $\hat{\sigma} = \sqrt{\text{MSW}}$ is the mean proportional between OS and O$\tilde{R}$; and the coefficient of determination SSB/SST is the ratio $\beta\bar{\bar{v}} : \left(\beta\bar{\bar{v}} + \bar{w}\right)$.



(a)



(b)

5. TWO-WAY ANOVA WITHOUT INTERACTION

Suppose that a randomized block design is used to assign treatments. Within each block, the same number of units – chosen randomly – are assigned to each treatment. Assume that there is no interaction between block effect and treatment effect. Then a two-way additive ANOVA model is applicable. This model tests the significance of treatment effect by first removing block effect from each response, and then applying a one-way ANOVA to the adjusted responses (after removing block effects). We illustrate the visualization of the F-statistic for testing treatment effect in a two-way additive ANOVA model using the data in example 4.

Example 4. Which of four methods of winding six strands of wire into a rope gives the best tensile strength (which is the heaviest load carried by the wire just before it breaks)? An experiment is conducted to measure the tensile strength of ropes made by each of four methods (a = 4) using wires from three different suppliers (b = 3). The experiment, being time consuming and destructive, is not replicated. The total sample size is $n = ab = 12$. Let $x_{i,j}$ (i = 1, 2, 3; j = 1, 2, 3, 4) denote the tensile strength of a rope made of wires from Supplier i using Method j. table 3 shows the data, and table 4 gives the results of a two-way ANOVA (without interaction) using R:

Table 3. Tensile strength of ropes made by four methods using wires from three suppliers

Tensile Strength	Method 1	Method 2	Method 3	Method 4
Supplier 1	70	67	61	66
Supplier 2	95	65	64	88
Supplier 3	93	69	70	80

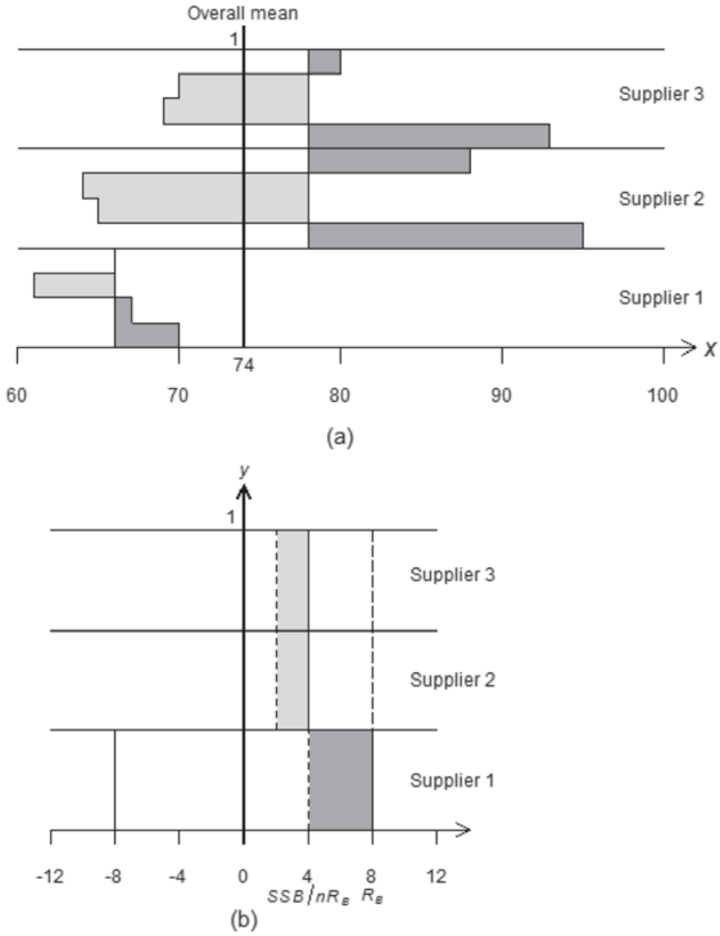
Table 4. Two-way ANOVA (without interaction)

Sources	Df	SS	MS	F	p-value
Supplier (Block)	2	384	192	3.84	.0844
Method (Treatment)	3	870	290	5.80	.0331
Residual	6	300	50	-	

We explain below how to visualize (some of) the statistics in the above two-way ANOVA table.

Before determining whether the four methods (levels of Factor A) result in significantly different tensile strength (response variable), we must first eliminate the variation due to supplier (block) effect. So, first we depict the tensile strengths as vertically stacked rectangles (of height $1/n$) of various widths, sorted by block. Using the vertical line method, we depict in figure 9(a) the block means and the overall mean. Note that the overall mean is not only the simple average of all measurements, but also the (simple) average of the block means (since the block sizes are equal). Next, we depict in figure 9(b) a scaled sum of squares due to blocks, SSB/(nR_B), where we choose R_B as the largest block deviation, as explained in section 4.

Figure 9. Responses within each block (supplier) yield block means, overall mean, block effects, block deviations, and scaled SSB, where the divisor R_B is the largest block deviation

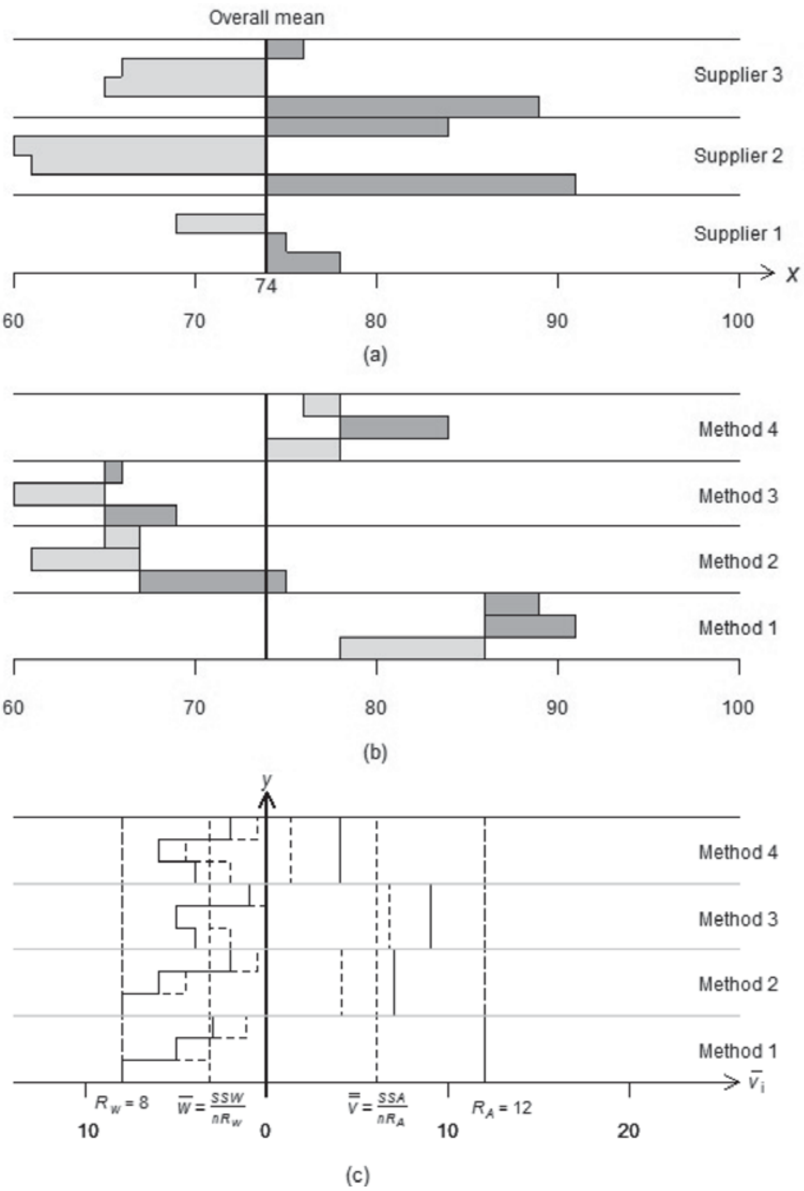


Thereafter, we remove the block effect (the within block mean tensile strength minus the overall mean) from each measurement. That is, we replace each tensile strength measurement $x_{i,j}$ by the block-adjusted measurement $y_{i,j} = x_{i,j} - \left(\bar{x}_i - \bar{\bar{x}}\right)$. Geometrically, it is equivalent to shifting all within block responses either left or right so that the corresponding blockmean-vertical-line falls exactly on the overall-mean-vertical-line. This we do for each block as shown in figure 10(a).

Next, in figure 10(b), we rearrange the block-adjusted responses—this time sorted by treatment (method)—in preparation for carrying out a one-way ANOVA to study the treatment (method) effect. The (block-adjusted) treatment means are shown in Figure 10(b). Note that the overall mean is still unchanged, since the total of all block effects is zero; that is, $\bar{\bar{y}} = \bar{\bar{x}}$.

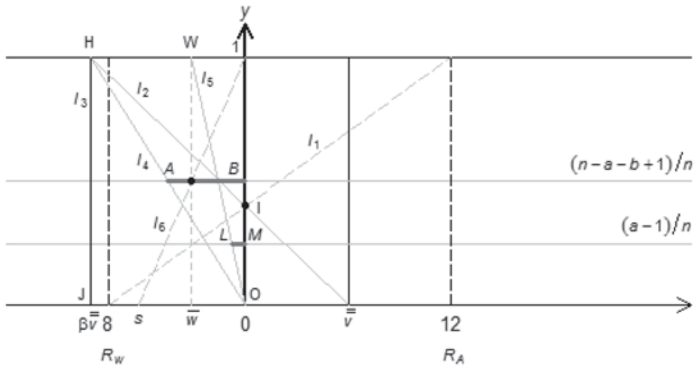
Following the method of section 4, in Figure 10(c), we show the absolute differences between treatment means and the overall mean, and a scaled treatment sum of squares due to Factor A or treatment (SSA/(nR_A)), where we choose R_A as the largest deviation of treatment mean from overall mean, to the right of the vertical axis. Simultaneously, to the left of the vertical axis in Figure 10(c), we show the absolute residuals $|e_{i,j}| = |y_{i,j} - \bar{y}_j|$ after removing the (block-adjusted) treatment means from the block-adjusted measurements. We show a scaled sum of squares of these absolute residuals or a scaled sum of squares within, $\bar{w}$ = SSW/(nR_w), where we choose R_W as the largest absolute residual.

Figure 10. (a) Block-adjusted responses (responses minus block effects), (b) Block-adjusted responses sorted by treatment groups, and (block-adjusted) treatment means, and (c) Deviations of treatment means from the overall mean (shown to the right of y-axis) yield $\bar{\bar{v}}$ = SSA/(nR_A), and deviations of block-adjusted responses from the treatment means (shown to the left of the y-axis) yield $\bar{w}$ = SSW/(nR_w). In all three panels, the vertical axis ranges over [0, 1]



Finally, the F-statistic for Factor A is shown in figure 11 as the ratio AB:LM. One can also see the standard error $\hat{\sigma} = \sqrt{MSW}$ as the mean proportional between OS and OR$_w$. Likewise, one can also visualize the F-statistic for block effect (though that may not be of primary interest).

Figure 11. Visualizing the *F*-statistic of factor A as AB:LM, in example 4



6. SUMMARY AND DIRECTIONS OF FUTURE RESEARCH

We recall a vertical line method that equalizes the areas of two sets of rectangles in order to visualize the mean, the MD, the MSD or the SD of a set of numbers. The method utilizes the ECDF *F* of the data, the ECDF *G* of the deviations and the ECDF *H* of the (scaled) squared deviations, all of which can be constructed using Euclidean plane geometry. The same Euclidean method helps us visualize a scaled SSB, a scaled SSW, and the (unscaled) F-statistic in a one-way ANOVA, as well as similar quantities in a two-way additive ANOVA model.

With some additional work we hope to extend the method to visualizing a two-way ANOVA *with* interaction, and also to visualizing the correlation coefficient and the least-squares regression line that study a linear relationship between two quantitative variables. We hope to report the details in future papers.

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